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RESEARCH STUDIES RELATED TO MAPPING, GEODESY AND POSITION DETERMINATION

FINAL TECHNICAL REPORT
19 December 1961 - 18 December 1962

Contract No. DA-44-009-ENG-3769
Project No. 8T35-12-001-01

M.R.I Project No. 2235-E

For

U. S. Army Engineer Geodesy, Intelligence and Mapping Research and Development Agency
Fort Belvoir, Virginia

MIDWEST RESEARCH INSTITUTE

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by

R. S. Brown
E. J. Martin, Jr.

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THE VIEWS CONTAINED HEREIN REPRESENT ONLY THE VIEWS OF THE PREPARING AGENCY AND HAVE NOT BEEN APPROVED BY THE DEPARTMENT OF THE ARMY.

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U. S. Army Engineer Geodesy, Intelligence and
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Fort Belvoir, Virginia

PREFACE

Work on a program to study the effects of natural phenomena on precise surveying and position determination methods and on the application of natural phenomena to geodesy was initiated on 19 September 1958 under Contract No. DA-44-009-ENG-3769 with the Topographic Systems Research Branch of the Topographic Engineering Department of the U. S. Army Engineer and Development Laboratories. On 30 January 1960, a supplemental agreement was executed which extended the period of the contract to 19 December 1961 and redirected the activities into narrower fields. On 26 January 1962, another agreement was executed which extended the period of the contract to 19 December 1963 and redirected the activities to the study of a particular distance measuring system. The work conducted after 19 December 1961 is summarized in this report.

During the current contract, the activity on the program has been under the direction of Mr. P. C. Constant, Head, Electronics Section; Mr. R. S. Brown has been the project leader, and is responsible for the theoretical work. Mr. E. J. Martin, Jr., has been responsible for the experimental designs.

Approved for:

MIDWEST RESEARCH INSTITUTE

Harold L. Stout

Harold L. Stout, Director
Engineering Division

14 January 1963

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SUMMARY

This project has been a feasibility study to determine whether it was possible to deduce the index-of-refraction profile along some path in the atmosphere from physical measurements made external to that path. This problem was solved and it was demonstrated that it was possible to make this remote measurement of the index-of-refraction profile using a quantity called the "transient reflection coefficient" of the atmosphere.

When an electromagnetic wave propagates into a medium of changing index-of-refraction, a certain portion of the incident energy is reflected back toward the source. The ratio of the reflected energy to the transmitted energy is called the reflection coefficient of the atmosphere. It was first shown that this reflection coefficient, which is a function of the frequency of the incident wave, is uniquely determined by the index-of-refraction profile. In addition, it was demonstrated that if the value of the reflection coefficient were known for all frequencies, then it would be possible to reconstruct the index-of-refraction profile.

Because the index-of-refraction profile changes with time, the reflection coefficient is also a time variable. Consequently, in order to determine the value of the reflection coefficient at all frequencies, it would be necessary to make a large number of simultaneous measurements each at a different frequency. Such a system, although feasible, would not be practical. It was because of this objection that the concept of a "transient reflection coefficient" was introduced.

When a very short pulse is transmitted into a region of variable index of refractions, the reflected energy will also be in the form of a pulse rather than a continuous wave. It was demonstrated that if the transmitted pulse were of a sufficiently short duration, then the ratio of the Fourier transforms of the reflected and transmitted pulses would contain sufficient spectral information to allow the construction of the index-of-refraction profile. It is this ratio that is called the "transient reflection coefficient" which differs from the ordinary reflection coefficient only in the way in which it is measured. The principal advantage of measuring the reflection coefficient by a transient method is that only a single measurement need be made. An additional advantage is that, because the complete measurement can be made in a very short interval of time, the adverse effects of the temporal variations of the index-of-refraction profile are nullified.

Experiments to check the utility of the inverse scattering theory are proposed, and discussed. These experiments consist of (a) setting up a transmission path whose length and index-of-refraction profile are accurately known,

(b) measurement of the reflection coefficient of the transmission path, (c) calculation of the index-of-refraction profile using data collected in (b), (d) calculation of the length of the transmission path from the experimentally determined index of refraction,and (e) the results of (d) will be compared with a true length of the line.

I. INTRODUCTION

This program has been a feasibility study of a new technique for using the transient reflection coefficient of the troposphere to make corrected distance measurements.

Present systems for making precise distance measurements transmit electromagnetic energy, such as light or radio waves, through the atmosphere to a reflector. The transmitted energy is then reflected back to the point of transmission and, by measuring the time necessary for the energy to make this transit, the distance between the transmitter and reflector is deduced. In all the present systems, it is assumed that the energy travels with a constant velocity, the value of which is determined by the local atmospheric conditions at each end of the survey path.

The index of refraction varies in a random manner along the survey path. These variations of the index of refraction cause corresponding variations in the velocity with which the electromagnetic energy propagates, and these variations cause errors in the indicated path length. If the value of the index of refraction were known everywhere along the survey path, then these errors could be accounted for and corrected. The task of this program has been to devise a method for measuring this index-of-refraction profile.

When a wave propagates into a medium of changing index of refraction, not only does the velocity of the wave change, but, in addition, a portion of the incident energy is reflected back toward the source. It has been shown* that the index-of-refraction profile can be deduced from a knowledge of the reflected energy. The demonstration of this fact represented the successful completion of task 3a as specified in the contract:

> "a. Conduct studies and investigations to determine the feasibility of determining the index of refraction of the troposphere between two points to be used for making a correction to a distance measurement made by the transmission of electromagnetic waves between the two points."

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* Brown, R.S., "Research Studies Related to Mapping, Geodesy and Position Determination," 14th Interim Technical Report, Contract No. DA-44-009-ENG-3769, 10 July 1962. Also see Appendices A and C, this report.

Subsequent work performed under this contract was concerned with task 3b, the second and final provision of the contract:

> "b. Contingent on the successful accomplishment of a. above, specify the system characteristics necessary to determine the refractive index of the troposphere between two points, e.g., transmitter power, pulse width, receiver sensitivity and noise figure, and computational circuitry."

With regard to task 3b, only a qualitative specification of the system characteristics could be made. Any further efforts toward system specification will have to be experimental because of (a) a lack of knowledge of the characteristics of the tropospheric index-of-refraction variations, and (b) the necessity of extending microwave techniques beyond the present state of the art.

II. INVESTIGATION

The system investigated under this contract is shown in Fig. 1a. A radar set transmits a wave, $f(t)$, into the turbulent region, i.e., into a region of variable index of refraction. As the transmitted wave, $f(t)$, traverses the turbulent region, reflections are produced and this reflected wave, $g(t)$, travels back to the radar set where it is detected. In the course of this investigation, it was shown that if both $f(t)$ and $g(t)$ were known, then $n(x)$, the index-of-refraction profile characteristics of the turbulent region, could be uniquely determined. Now, if two dielectric markers consisting of sheets of lucite, teflon, or other dielectric material were introduced into the propagation path, as shown in Fig. 1a, then they would produce two characteristic marks in the index-of-refraction profile as shown in Fig.1b and would serve to mark the beginning and end of the survey path. Thus, the distance between the dielectric markers would be determined as the difference between the abscissal values of the dielectric marks shown on the index-of-refraction profile.

A. Remote Measurement of the Index-of-Refraction Profile

A wave, $w(k,x)$, propagating in a medium of changing index of refraction, $n(x)$, satisfies the differential equation

$$\ddot{w}(k,x) + k^2n^2(x)w(k,x) = 0 \quad , \tag{1.1}$$

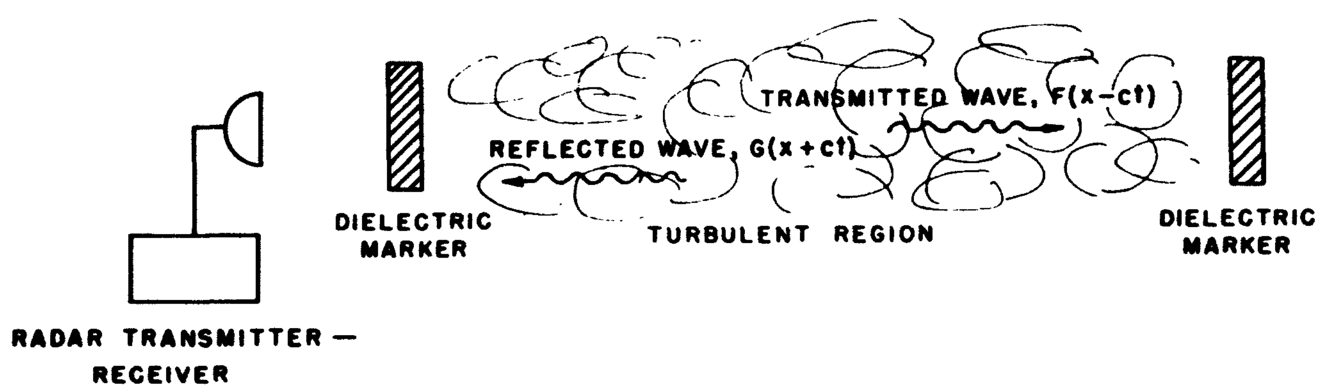

Fig. 1a - System to Measure Index-of-Refraction Profile

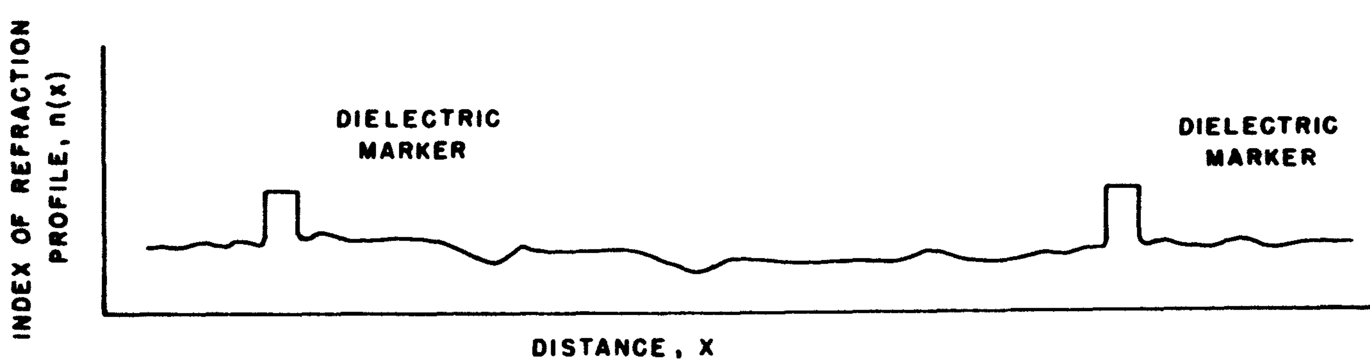

Fig. 1b - Measured Index-of-Refraction Profile

where k is the free space wave number of the wave and is related to the temporal frequency, ω , of the wave, and the free space (in vacuum) velocity of the wave, c , by

$$k = \omega/c \quad . \tag{1.2}$$

In order to determine $n(x)$, the equivalent equation,

$$\ddot{u}(k,x) + \left[k^2 - V(x)\right] u(k,x) = 0 \tag{1.3}$$

is first considered. Here, the term $V(x)$, called the scattering potential, is equivalent to $n(x)$.

Now, assume that

$$V(x) = 0 \quad \text{for} \quad x < 0 \tag{1.4}$$

and that

$$u(k,x) = e^{ikx} + r(k)e^{-ikx} \quad \text{for} \quad x \le 0 \quad , \tag{1.5}$$

where the term $r(k)$, the reflection coefficient, is completely known.

In Appendix A of this report, it is shown that $r(k)$ is an element of a certain operator, S , called the "Scattering Matrix":

$$S = \begin{pmatrix} r(k) & \tau(k) \\ t(k) & \rho(k) \end{pmatrix} \quad . \tag{1.6}$$

It is shown that the elements of this matrix, which are functions of the free space wave number k only, are uniquely determined by the scattering potential, $V(x)$.

Fortified with the knowledge of the uniqueness of S , the research efforts were then directed toward the solution of the inverse scattering problem, namely, the problem of determining $V(x)$ from a knowledge of S . This problem was solved in terms of a function $K(x,y)$ defined by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{u(k,x)\hat{v}(k,y)}{|t(k)|^2}\,dk = K(x,y) + \delta(x-y) \tag{1.7}$$

(Eq. (A-41) of Appendix A), where $t(k)$ is the transmission coefficient (another element of S) and $\hat{v}(k,y)$ is a certain normalized wave (see Eq. (A-38) et seq. of Appendix A). It was further shown that $V(x)$ and $K(x,y)$ were related by

$$V(x) = 2\,\frac{d}{dx}\,K(x,x) \tag{1.8}$$

(see Eq. (A-55) of Appendix A), and that $K(x,y)$ could be found by solving the integral equation

$$0 = R(x+y) + K(x,y) + \int_{-x}^{x} R(y+z)K(x,z)dz \quad , \tag{1.9}$$

where

$$R(x+y) = \frac{1}{2\pi}\int_{c} r(k)e^{-ik(x+y)}\,dk \tag{1.10}$$

(see Eq. (A-57) et seq. of Appendix A).

Equation (1.9) represents a solution of the inverse scattering problem. It was shown that this solution would be valid whenever certain conditions pertaining to $R(x+y)$ were met (see conditions (1) to (5), p. 42, Appendix A and also Eqs. (A-60) to (A-71), Appendix A).

The next task was to determine a method whereby the index-of-refraction profile, $n(x)$, could be constructed from the scattering potential $V(x)$. This construction is given in Section IV of Appendix A and consists of the following four steps:

1. Find $U(z)$ from

$$\ddot{U}(z) - V(z)U(z) = 0 \quad , \tag{1.11}$$

2. Find $x(z)$ from

$$x(z) = c\int \frac{dz}{U^2(z)} \quad , \tag{1.12}$$

3. Invert $x(z)$ to obtain $z(x)$, and

4. Find $n(x)$ from

$$n(x) = U^2(x) \quad . \tag{1.13}$$

This construction completes the inverse scattering problem. The next task was to investigate methods whereby $r(k)$ could be measured.

B. The Measurement of the Reflection Coefficient, $r(k)$

The solution of the inverse scattering problem given in Section II-A was based upon the assumption that $r(k)$, the reflection coefficient, was known. In Appendix B, a theory is presented for the measurement of $r(k)$, and it is upon this theory that the proposed experimental system to measure the index of refraction is based.

Very briefly, a forward traveling wave, $F(x-ct)$, will be transmitted into the region of nonconstant index of refraction and a backward traveling, reflected wave, $G(x+ct)$, will be produced. Both of these waves will be detected and recorded. Then the quantities $f(k)$ and $g(k)$ will be computed,

$$f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x-ct) e^{-ik(x-ct)} d(x-ct) \qquad (1.14)$$

and

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(x+ct) e^{ik(x+ct)} d(x+ct) \quad , \qquad (1.15)$$

and, from these, $r(k)$ can be found from

$$r(k) = \frac{g(k)}{f(k)} \qquad (1.16)$$

(see Eqs. (B-11, B-12, B-13), Appendix B).

C. An Alternate Derivation of the Inverse Scattering Theory

The representation of the reflection coefficient, $r(k)$, as the ratio of the spectrum of the reflected wave to that of the transmitted wave led to an alternate derivation of the inverse scattering theory in which the directly measurable terms, $f(k)$ and $g(k)$, were used instead of their ratio. This alternate derivation is presented in Appendix C, and follows exactly the same course as the first derivation except that the wave, $u(k,x)$, was initially normalized as

$$u(k,x) = f(k)e^{ikx} + g(k)e^{-ikx} \quad \text{for} \quad x \le 0 \quad , \qquad (1.17)$$

instead of as in Eq. (1.5). With this normalization, $K(x,y)$ appears at the solution of the integral equation

$$0 = G(y+x) + \int_{-x}^{x} K(x,z)F(y-z)dz + \int_{-x}^{x} K(x,z)G(y+z)dz \qquad (1.18)$$

instead of

$$0 = R(x+y) + K(x,y) + \int_{-x}^{x} K(x,z)R(y+z)dz \quad . \qquad (1.19)$$

In Eq. (1.18), the functions $F(x)$ and $G(x)$ are the Fourier transforms of $f(k)$ and $g(k)$, respectively.

The two inversion formulae, Eqs. (1.18) and (1.19), are equivalent in all respects. The primary difference between them lies in the fact that Eq. (1.18) utilizes directly measurable data, whereas the function $R(x+y)$ appearing in Eq. (1.19) is the result of several mathematical operations performed on $F(x)$ and $G(x)$.

Two examples are presented in Appendix C which serve to indicate the equivalence of the two inversion formulae. In the first example, the functions

$$f(k) = \frac{i(k-i)}{(k+i)^2} \tag{1.20}$$

and

$$g(k) = \frac{-i}{(k+i)^3} \tag{1.21}$$

were selected. Then

$$r(k) = \frac{g(k)}{f(k)} = \frac{-1}{k^2+1} \quad . \tag{1.22}$$

It was then shown that both integral equations lead to the same differential equation,

$$0 = K_{yy}(x,y) - K(x,y) + K(x,-y) \quad , \tag{1.23}$$

whose solution is (taking into account the boundary conditions, see Eq. (C-30), Appendix C)

$$K(x,y) = -\frac{\sinh\sqrt{2}x + \sinh\sqrt{2}y}{\sqrt{2}\cosh\sqrt{2}x} \quad . \tag{1.24}$$

From this value of $K(x,y)$, it follows that

$$V(x) = -4 \sinh^2 \sqrt{2}x \quad . \tag{1.25}$$

In the second example, the functions

$$f(k) = \frac{i}{k+i} , \tag{1.26}$$

$$g(k) = \frac{1}{(k+i)^2} \quad , \tag{1.27}$$

and

$$r(k) = \frac{g(k)}{f(k)} = \frac{-i}{k+i} \tag{1.28}$$

were selected. In this case, both integral equations lead to the differential equation

$$0 = K_y(x,y) + K(x,y) + K(x,-y) \quad , \tag{1.29}$$

whose only solution is

$$K(x,y) = \text{constant} \quad . \tag{1.30}$$

Thus, in the second example,

$$V(x) = 0 \quad . \tag{1.31}$$

D. Calibrated Transmission Path

To check the utility of the inverse scattering theory just described, it is proposed that certain physical experiments be performed. These experiments will consist of setting up a transmission path whose length and index-of-refraction profile are accurately known. Next, the reflection coefficient of the transmission path will be measured and, from these data, the index-of-refraction profile will be calculated with the methods of either Appendix A or Appendix C. Finally, the length of the transmission path will be calculated from the experimentally determined index-of-refraction profile, and this result will be compared with the true length of the line to obtain an estimate of the accuracy which can be expected from a distance measuring system based on the principles presented in this report.

The calibrated transmission path will consist of a section of K_u band waveguide about 1 meter long. Associated equipment will consist of a microwave oscillator covering the 10 to 20 Gc. band of frequencies and necessary reflection measuring devices (a complete listing of the necessary equipment is given in Appendix D). With this equipment, the length of the line can be determined to an accuracy of ± 1 mm. (when the index of refraction is constant).

To perturb the index-of-refraction profile, blocks of dielectric material such as teflon will be introduced into the waveguide. This will cause the electrical length of the line to increase and will also cause a certain portion of the electromagnetic energy propagating within the line to be reflected. These reflections can be measured (on a steady-state basis) by standard techniques, for instance, with a slotted line or with a Hewlett-Packard 416A ratio meter.

In a more practical situation, i.e., for measuring the index-of-refraction profile of a turbulent atmosphere, these steady-state techniques will fail and a transient method must be used. As the term is used here, a transient measurement is one in which the complete measure of the index-of-refraction profile is made during an interval which is short with respect to the rate of change of the index-of-refraction profile. To do this, a very short impulse of electromagnetic energy will be transmitted into the turbulent medium and a corresponding reflected impulse will be received. Then, if both the transmitted and reflected impulses are known, the reflection coefficient can be determined with the methods of Appendix B.

E. Radio Frequency Energy Source

Methods for obtaining the short pulses (of the order of a few nanoseconds in duration) of radio frequency (rf) energy that will be necessary for the actual index-of-refraction profile measurements have been investigated. Among several approaches that have been considered, a method based on the use of ordinary radar pulse techniques with subsequent shaping of the rf pulse has been selected as most promising. This method contemplates the use of a standard pulser, a magnetron, and a rather unconventional application of a gas-filled, transmit-receiver (T-R) tube as a pulse-shaping device.

Work on the radio frequency energy source has been concerned with the specification of only (a) a magnetron and a suitable pulser (modulation equipment), and (b) the design of appropriate control and monitoring equipment. The tentative decision to perform the experiments of K_u-band frequencies led to the selection of a Type QK 319 magnetron. Past experience has shown that this tube can be successfully driven by an APS-2 hard-tube radar modulator. Design of control circuitry has included means for selecting any one of nine crystal-controlled pulse repetition frequencies between approximately 250 pps and 1,650 pps. Means for selecting any one of three pulse durations (0.5, 1.0 and 2.0 μsec.) are also provided. For simplicity and flexibility, it is contemplated that the radar modulator and associated control circuitry will be mounted in a single, portable unit. This portable modulator unit will also contain a Tektronix RM 16 oscilloscope for monitoring the operation of the system.

The design of the radio frequency energy source is discussed in detail in Appendix E. In addition, a pulse shaper to form nanosecond pulses from the microsecond pulses generated by the radio frequency energy source will be required. This component is discussed in Section III-B.

III. DISCUSSION

The problem of accurately measuring distances is of fundamental importance in the science of mapping, geodesy, and position determination. Variations in the process of determining the position of a point "A" with respect to another point "B" can be obtained through application of different trigonometric relations. But position determination must ultimately depend upon either distance measurements or a combination of distance and angle measurements.

For centuries, the arts of surveying and mapping were completely dependent upon some form of the level or transit for measuring angles and upon some form of the chain or tape for making distance measurements. While proper use of these devices can result in highly accurate determination of the position of one point with respect to another, the process is a slow one, especially when the position determination is to be carried out over relatively large distances on the earth's surface. It has long been the desire of surveyors and map makers to be able to accomplish position determination, over long distances, rapidly and accurately. The development of radar and similar electronic devices during World War II provided a great step toward the realization of this goal.

Since 1945, a number of military equipments have been tried in precise, long-distance, measurement experiments. In addition, a number of devices or systems have been developed specifically for the purpose of accurately measuring long distances. Most of the members of this "first generation" of distance-measuring instruments depend on the propagation of electromagnetic energy, such as light or radio waves, through the atmosphere to a reflector. A portion of the transmitted energy is reflected back to the point of transmission, and the distance between the transmitter and the reflector is deduced from the time required for the energy to make this transit.

In all existing systems of distance measurement by means of electromagnetic propagation, it is assumed that the energy travels with a constant velocity, the value of which is determined from the local atmospheric conditions at each end of the survey path. In reality, the index of refraction can vary in a random manner along the survey path. Such variations of index of refraction cause corresponding variations in the velocity of propagation of the electromagnetic wave. These variations in propagation velocity can produce errors in the indicated path length. The only sure means of eliminating such errors depends on a knowledge of the index-of-refraction profile along the entire transmission path.

The random variations of the index of refraction cause another effect, namely, the reflection of a part of the incident electromagnetic energy back toward the source. It was the purpose of this project to determine if this reflected energy could be utilized to obtain more precise distance measurements. The major result of the project was the theoretical demonstration of the fact that the reflected energy carried with it sufficient information to construct the index-of-refraction profile along the survey path.

A. Determination of the Index-of-Refraction Profile

The starting point for the determination of the index-of-refraction profile was the Schrödinger wave equation

$$\ddot{u}(k,x) + \left[k^2 - V(x)\right] u(k,x) = 0 \quad . \tag{1.3*}$$

In this equation, the term $u(k,x)$ represents a wave of spatial frequency or wave number k, propagating in the x direction. The term $V(x)$ is called the scattering potential and represents the variations in the energy (or velocity) of the wave. By assuming that $V(x) = 0$ for $x < 0$, it was possible to write a partial solution of Eq. (1.3) as

$$u(k,x) = Ae^{ikx} + Be^{-ikx} , \; x \le 0 \quad .$$

In this representation, the term $A\exp(ikx)$ represents a wave propagating from $x = -\infty$ into the region where $V(x) \neq 0$ and the term $B\exp(-ikx)$ represents a wave reflected from the variations of $V(x)$. This identification is consistent with the assumption of a suppressed temporal factor $\exp(-i\omega t)$.

The terms A and B represent the magnitude of the transmitted and reflected waves, respectively. In general, both A and B are functions of frequency, k, and are related by the equation

$$|A|^2 \ge |B|^2 \quad ,$$

which is a statement of the law of conservation of energy. However, because $|A|^2$ is a measure of the transmitted power, A may be arbitrarily selected as being $A = 1$. In this case, the partial solution of the Schrödinger equation takes the special form

$$u(k,x) = e^{ikx} + r(k)e^{-ikx} , \; x \le 0 \quad . \tag{1.5}$$

* Note that the equation numbers used in this section are identical to the numbers used on the corresponding equations in Section II.

In Eq. (1.5), the term $r(k)$ is called the reflection coefficient; it is a measure of the amount of energy which is reflected. It is also an element of a certain matrix, S, called the scattering matrix:

$$S = \begin{pmatrix} r(k) & \tau(k) \\ t(k) & \rho(k) \end{pmatrix} . \tag{1.6}$$

In Appendix A, the significance of S is discussed and a number of mathematical expressions relating its elements are derived.

Once the scattering matrix and its relationship to the scattering potential had been defined, it was possible to give a concise mathematical statement of the problem: If $r(k)$ is known, can $V(x)$ be found? This problem was solved by presenting a method whereby $V(x)$ could be uniquely constructed from $r(k)$.

It is a well-known fact that both $u(k,x)$ and $\dot{u}(k,x)$ are continuous functions of x. Thus, since $u(k,x)$ was known for all negative values of x, it was only necessary to determine a method whereby this known function could be extrapolated into the space $x \geq 0$ in such a way that $u(k,x)$ and $\dot{u}(k,x)$ would be continuous and such that

$$\ddot{u}(k,x) = \left[V(x)-k^2\right]u(k,x) \quad .$$

This extrapolation was performed by introducing two new functions, $v(k,x)$ and $K(x,y)$. The first of these functions, $v(k,x)$, was defined as the wave which would (a) be equal to $u(k,x)$ for $x \leq 0$, and (b) would satisfy the Schrödinger equation for $x \geq 0$ if $V(x)$ were equal to zero. The second function, $K(x,y)$, was not explicitly defined; it appeared as the kernel of the integral transform

$$u(k,x) = v(k,x) + \int_{-x}^{x} K(x,y)v(k,y)dy \tag{a}$$

(see Eq. (A-46), Appendix A). The use of this extrapolation formula is a rather standard technique in the theory of partial differential equations. It was known that $u(k,x)$ obtained from Eq. (a) satisfied all of the required boundary conditions (this fact is demonstrated in Appendix A); it was necessary,

however, to demonstrate that a relationship did exist (a) between $K(x,y)$ and $V(x)$, and (b) between $K(x,y)$ and $r(k)$. This demonstration is presented in Appendix A, Eq. (A-47) et seq. Very briefly, the results are:
(a) $K(x,y)$ is related to $V(x)$ by

$$V(x) = 2 \frac{d}{dx} K(x,x) \quad , \tag{1.8}$$

and (b) $K(x,y)$ is related to $r(k)$ by

$$0 = R(x+y) + K(x,y) + \int_{-x}^{x} R(y+z)K(x,z)dz \tag{1.9}$$

where

$$R(x+y) = \frac{1}{2\pi} \int_{c} r(k)e^{-ik(x+y)} \, dk \ . \tag{1.10}$$

In other words, if $r(k)$ were known, then $K(x,y)$ could be found from Eq. (1.9) and $V(x)$ from Eq. (1.8).

Attention is now turned to the problem of determining the index-of-refraction profile, $n(x)$. If $u(k,x)$ represents an electromagnetic wave propagating in the atmosphere, then it satisfies not the Schrödinger equation, but rather the Helmholtz equation:

$$u(k,x) + k^2n^2(x)u(k,x) = 0 \quad .$$

However, the effort involved in finding $V(x)$ was worthwhile. In Appendix A, Eq. (A-72) et seq., it is shown that if $V(x)$ is known, then the corresponding $n(x)$ can be found. This fact is demonstrated by a standard, but rather involved, change of variable techniques; the reader is referred to Appendix A, Eqs.(A-72) to (A-92) for the details.

The next task in this study was concerned with the method whereby $r(k)$ could be measured. This problem is discussed in Appendix B where it is

shown that if a short impulse, $F(x-ct)$, is propagated into the medium of varying index of refraction and if a reflected pulse, $G(x+ct)$, is received, then $r(k)$ can be found from the equation

$$r(k) = \frac{g(k)}{f(k)} \tag{1.16}$$

where

$$f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty} F(x-ct) e^{-ik(x-ct)} d(x-ct) \quad , \tag{1.14}$$

and

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty} G(x+ct) e^{ik(x+ct)} d(x+ct) \quad . \tag{1.15}$$

B. Proposed Experimental Verification of the Theory

On the basis of the theoretical results obtained under this program, it is possible to deduce the index-of-refraction profile from measurements of the reflections. However, because it is always desirable to have experimental evidence to support a theory, an experimental research program to obtain these data has been designed. The objectives of this program will be:

1. To verify the theory presented in this report;

2. To design and assemble an experimental system, based on established theory, which can be used to accurately determine the index-of-refraction profile along a prescribed transmission path; and

3. To develop the techniques of using this system for the precise measurement of distance.

To achieve these objectives, it will be necessary to accomplish the following tasks, in essentially the order listed:

1. Establish a calibrated transmission path for electromagnetic waves, along which the index of refraction can be precisely controlled and (within limits) changed at will.

2. Design and fabricate a radio frequency energy source capable of generating, on a repetitive as well as a triggered or single-shot basis, very short pulses (of the order of a few nanoseconds in duration) of electromagnetic energy at a relatively high peak power level.

3. Design and fabricate a signal separation system capable of sampling the pulses transmitted by the system described in (1) and the reflections of those signals from the calibrated transmission path described in (2).

4. Design and fabricate a monitor system capable of measuring pertinent characteristics of the direct and reflected signals produced by the signal separation system described in (3).

5. Establish data processing techniques suitable for handling the experimental data obtained from the monitoring system described in (4), thereby obtaining the index-of-refraction profile along the calibrated transmission path.

6. Perform a sufficient number of experiments, using the combination of equipment and techniques described in (1) through (5), to establish the feasibility of accurately measuring distance by this method.

The calibrated transmission path mentioned in task 1 has been designed and the major components are listed in Appendix D. The radio frequency energy source (task 2) has been partially designed and is reported in Appendix E. Further design must be made on an experimental or cut-and-try basis because the requirements outlined in tasks 1 through 4 are beyond the present state-of-the-art of microwave technology.

The proposed research program could, theoretically, be carried out using either optical or radio frequency electromagnetic energy. However, there are certain factors which make microwaves a much more attractive choice than light for the purposes of this investigation. First, the signal that will be reflected from regions of varying index of refraction on the calibrated transmission path will be only a small fraction of the incident signal; preliminary analysis indicates that reflections will be about 100 db below incident signal power levels. Consequently, quantitative reflected signals would be extremely difficult to measure if the transmitted signals were not incident on the transmission path at relatively high power levels. Recent

advances in laser technology have resulted in devices capable of producing well-collimated, high-intensity beams of light, but techniques for obtaining very short (nanosecond) pulses of laser output have not been fully developed. Pending advances in the laser field may alter this situation in the near future, but microwave techniques which offer a great deal of promise obtaining short-duration pulses of rf energy are in existence. Second, even if short pulses of high-intensity light could be produced, practical limitations on the power levels of incident signals that could be handled safely in the proposed experimental system would result in reflected signals so weak as to be difficult to detect by any known optical method. This is not believed to be the case for microwave signals because amplification techniques are highly developed in the field of microwave electronics.

On the basis of the foregoing considerations, it is contemplated that the proposed experimental research program will be carried out using radio frequency (microwave) electromagnetic waves. The exact microwave frequency that will be employed depends on a number of factors. The shorter wavelengths of the higher microwave frequencies offer the promise of greater inherent accuracy in distance measurement. However, the variety of commercially available components and equipment suitable for use above the K-band frequencies (18.0-26.5 Gc.) is very limited. Therefore, K-band is considered optimum for the proposed program. Other frequencies could be used, with X-band being the lower limit.

On the other hand, if equipment which would operate in the millimeter wavelength range were to become commercially available, it would be interesting to extend the range of investigation up to, perhaps, 130 Gc. to evaluate more carefully the effects of atmospheric attenuation. A curve of attenuation versus frequency is shown in Fig. 2.

Regardless of the microwave frequency that is finally selected for the performance of the experiments, the general nature of the experimental system will be as shown in Fig. 3. The shaded portions of this figure indicate those portions of the system that can be easily established with present-day technology. Although some effort will be required to procure, assemble, and adjust these portions of the system, no major technical problems are anticipated in these areas. The major part of the research effort on the proposed program will be devoted to the design and assembly of those portions of the system shown by the unshaded blocks: (1) the pulse shaper in the rf energy source, (2) the signal separation system, and (3) the amplifier in the monitor system.

The calibrated transmission path will consist of a section of waveguide several hundred wavelengths long at the selected operating frequency.

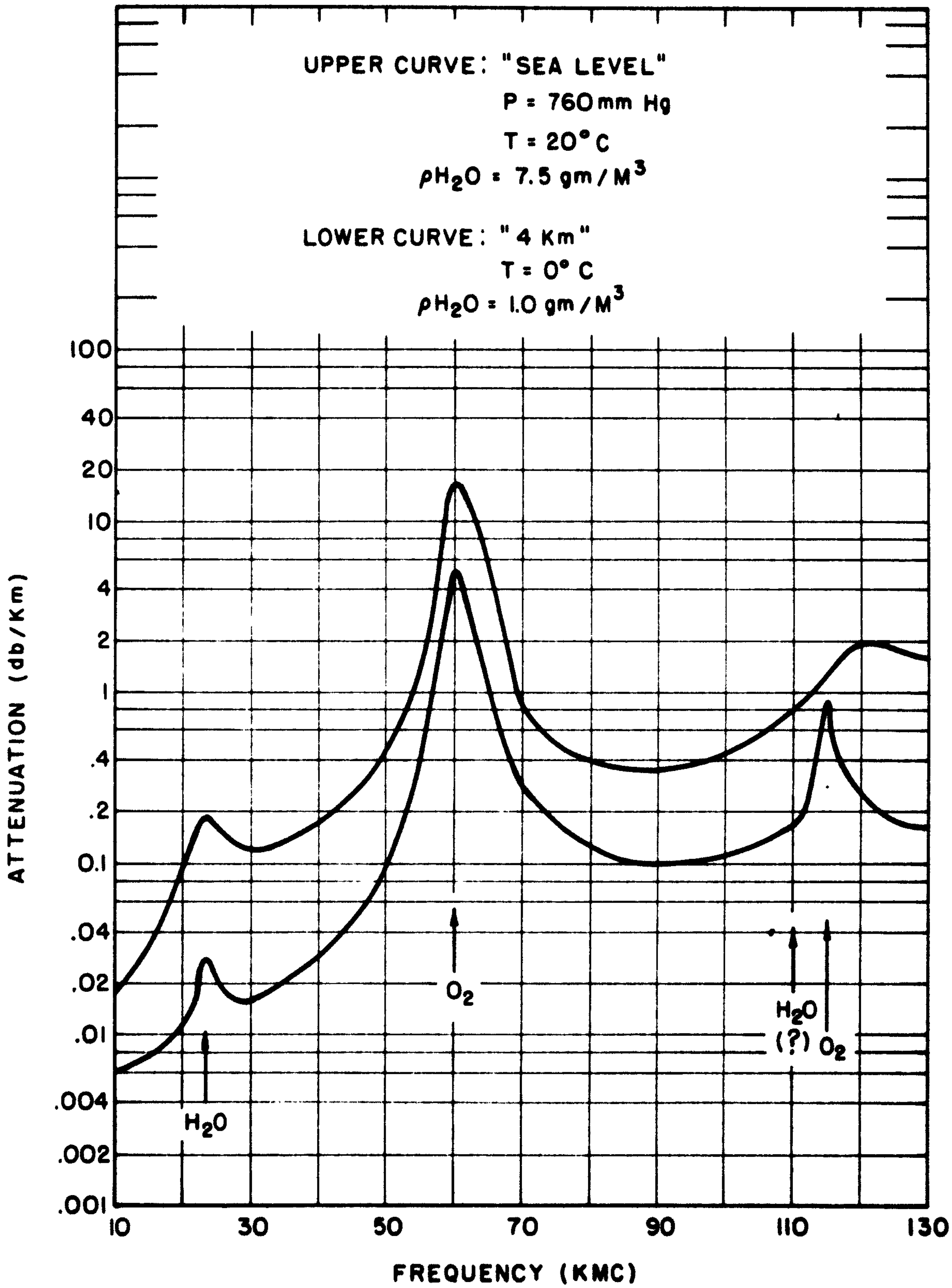

Fig. 2 - Attenuation Characteristics of the Atmosphere

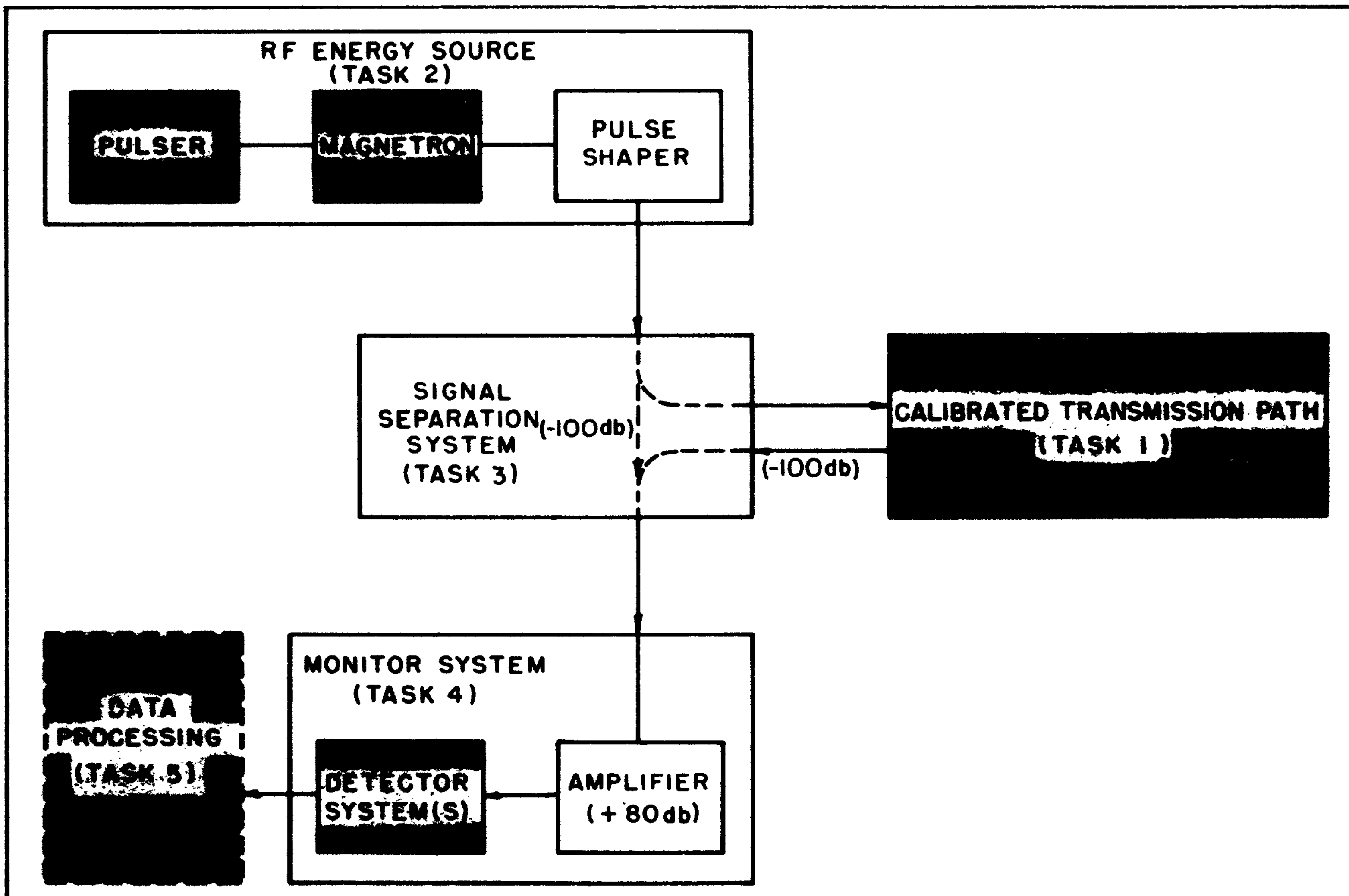

Fig. 5 - Block Diagram of Proposed Experimental System

An additional 15-30 ft. of waveguide will be used to produce a delay in the reflected signal so that switching operations can be performed in the signal separation system. The remainder of the waveguide path will be sealed and evacuated to provide a constant, stable atmosphere in which the actual distance measurements will be made. However, provisions will be made for opening the evacuated line so that dielectric perturbations of known characteristics can be introduced into the waveguide at various places. The entire waveguide system will be set up on a stable foundation which will include provisions for leveling and accurately aligning the path.

For actual index-of-refraction profile measurements, the tunable, c-w sources used in the calibration of the transmission path will be replaced by a pulsed source. This pulsed source will consist of a magnetron tube and associated modulation circuitry for producing pulses of rf energy either on a single-shot or repetitive basis, as desired. To obtain the extremely short pulses (of the order of a few nanoseconds in duration) necessary for the proposed experiments, additional microwave circuitry will be necessary in the form of a pulse-shaping system at the output of the magnetron. The development of this pulse-shaping system represents one of the major areas of research effort on the proposed program.

An approach to the solution of the pulse-shaping problem is suggested in Fig. 4. Here, a fast-ionizing gas tube is used to limit the transmitted signal to a "leakage spike" which is only a few rf cycles in duration; a ferrite circulator device provides protection for the magnetron. The use of a secondary-emission "multipactor" device rather than a gas "transmit-receive" (T-R) tube presents an alternate approach to the solution of the pulse-shaping problem. Certain new types of voltage-controlled ferrite or solid-state waveguide switches offer additional promising approaches to the solution of this problem.

Since the determination of the index-of-refraction profile depends on a knowledge of the signal incident upon the transmission path, as well as the signal reflected from variations in the index of refraction along the path, a means for monitoring both incident and reflected signals is an essential portion of the experimental system. The fact that the reflected signal will, very likely, be quite small makes the separation of incident and reflected signals rather complicated. Separation of the signals must be accomplished with approximately 100 db attenuation of the incident pulse and practically no attenuation of the reflected signal. The development of such a signal separation system represents a second major area of research on the proposed program.

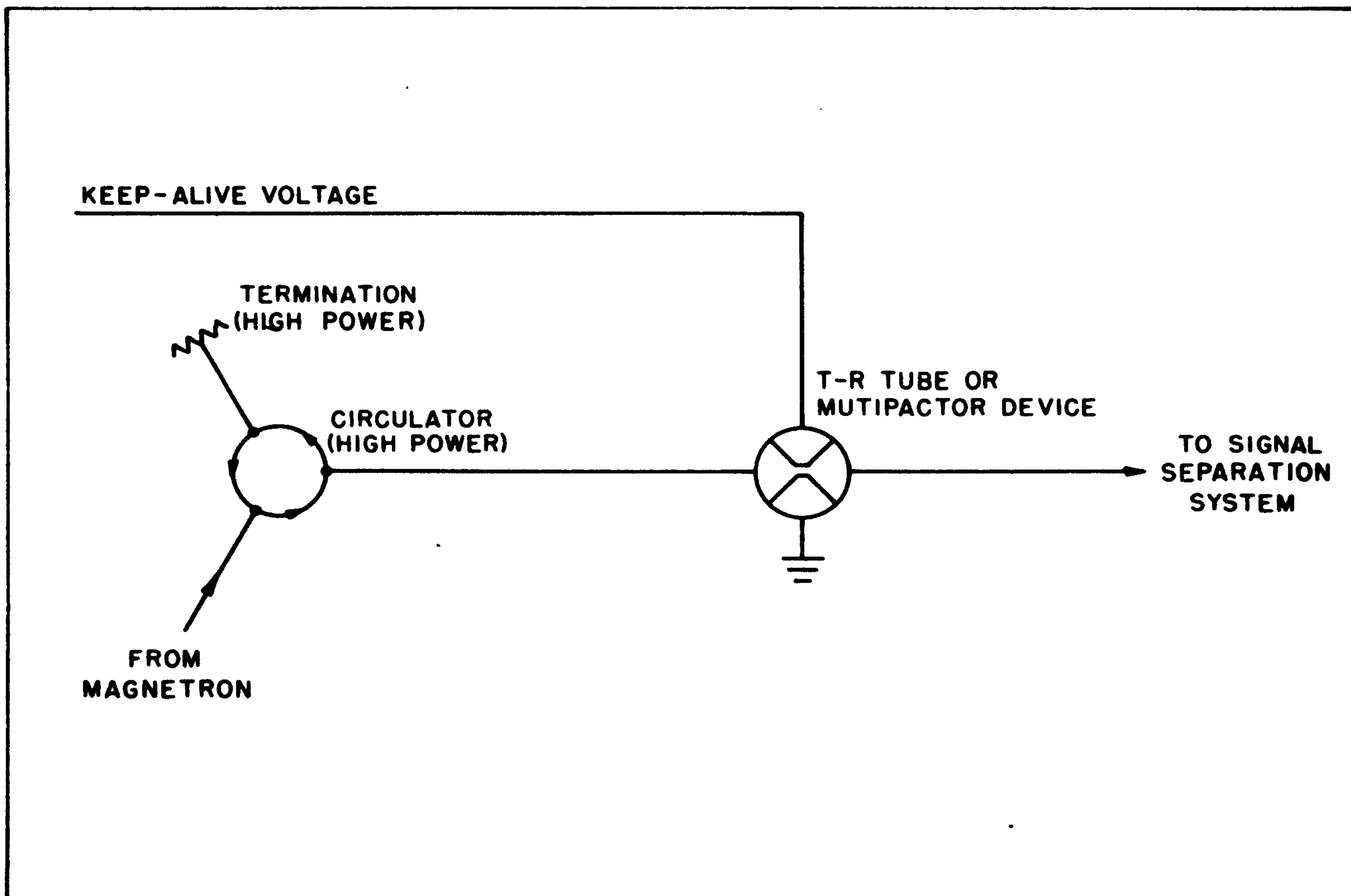

Fig. 4 - Radio Frequency Pulse Shaper

A possible approach to the solution of the signal separation problem is presented in Fig. 5. The ferrite circulator "1" allows the signal from the pulse shaper to be transmitted to the calibrated transmission path with practically no attenuation in the forward direction (the direction of the arrows). Leakage in the reverse direction through this circulator results in approximately 20 db of attenuation of the incident signal that reaches the ferrite circulator "2". The major portion of the signal which reaches circulator "2" is transmitted to the termination "A". Leakage in the reverse direction through circulator "2" results in an additional 20 db of attenuation of the incident signal which reaches circulator "3". The process is repeated through circulators "4" and "5" to provide approximately 100 db of attenuation of the incident signal which is ultimately transmitted to the monitor system. After a sufficient delay to allow the incident pulse to reach the monitor system, a striking voltage is applied to the four T-R tubes causing them to ionize. If the section of waveguide between the first ferrite circulator and the calibrated transmission path is sufficiently long, the T-R tubes will be fully ionized by the time the reflected signal returns from the transmission path, and the four terminations will be essentially replaced by short circuits. The reflected signal will be readily transmitted from the circulator "1" to the circulator "2", to the short circuited section, back to circulator "2", to circulator "3", and so on through the entire signal separation system (with only a few decibels of attenuation) to the monitor system.

Both the incident and reflected signals which reach the monitor system will be small. Consequently, the first portion of the monitor system must provide amplification of the signal. If it is assumed that output of the pulse shaper comprises 1.0-nanosecond pulse of 50 kw. peak power at 1,000 pulses/sec, it is readily seen that the average power output of the rf source is about 50 mw. Thus, about 80 db of the 100 db attenuation introduced by reflection or signal separation or both must be restored by an appropriate amplifier if a conveniently detectable 0.5 mw. signal level is to be maintained at the detector system input. The assembly of a suitable amplifying system is the third major area of research effort associated with the proposed program. Traveling wave tubes and parametric amplifiers offer promise in meeting the requirements for restoration of the 100 db attenuation.

The remainder of the monitor system comprises instrumentation for measuring pertinent characteristics of both the transmitted and reflected signals. Amplitude versus time information, obtained with a nanosecond sampling oscilloscope, and power level measurements, obtained with a bolometer and rf power meter, will be necessary for the determination of the index-of-refraction profile along the path. It is contemplated that commercially available instruments will suffice for the collection of these data, at least

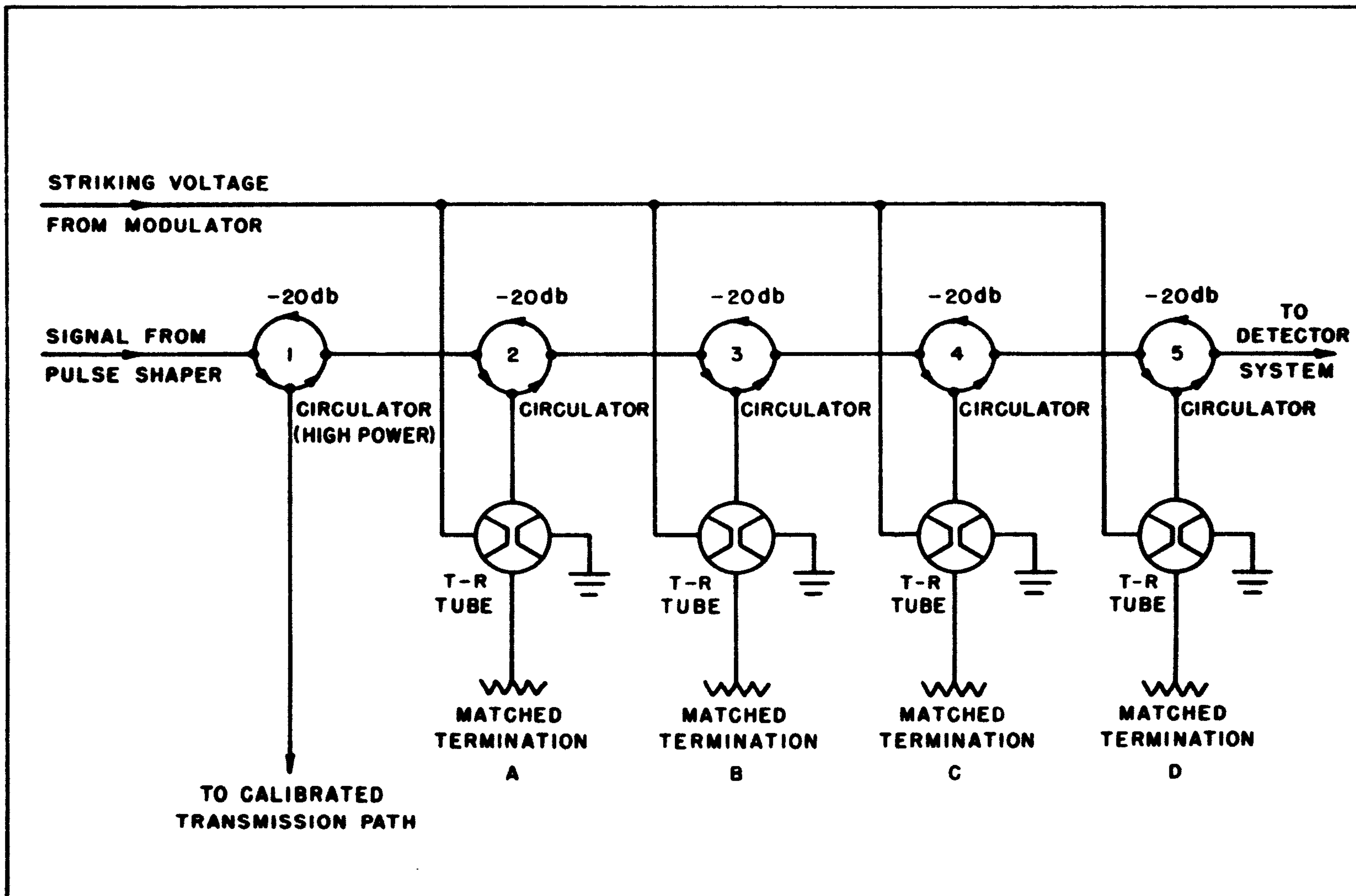

Fig. 5 - Signal Separation System

in early phases of the program. Modifications of commercial equipment or the development of specialized instrumentation may become necessary in later phases of the program.

Processing of experimental data will be accomplished manually or with the help of Midwest Research Institute's computer facilities. Although automatic data processing will be desired at some future time, practically no effort in the development of a special data processing system is contemplated in connection with this program. Since the data processing requirements are well defined by the theory which forms the basis of the proposed experimental program, only a small amount of project effort will be necessary to establish the computational procedure.

Experiments will actually begin with the testing of the various systems and combinations of systems as they are developed. However, once the over-all experimental system has been perfected, efforts will be directed toward the perfection of techniques for operation of the system in the precise measurement of distance. Also,experimental evidence of the system's over-all capabilities will be gathered.

IV. CONCLUSIONS

1. The reflection coefficient, $r(k)$, associated with an index-of-refraction profile, $n(x)$, can be determined from the spectral interpretation of two impulses of electromagnetic energy.

2. If $r(k)$ is known, then $n(x)$ can be computed.

3. Experimental verification of the theory presented in this report should be pursued.

V. OTHER APPLICATIONS OF THE THEORY

Although the theory presented in this report was developed for the purpose of obtaining highly accurate distance measurements, it has a number of other far-reaching applications. Basically the theory indicates the method whereby the characteristics of a medium in which a wave can propagate can be measured remotely and rapidly and without appreciably disturbing the medium. For instance, not only could the spatial characteristics of the index of refraction be measured but, by making a sequence of such measurements, the

temporal structure of the turbulence could also be investigated. An investigation such as this would provide data invaluable to, for instance, the study of energy transfer in turbulent flow.

Other applications of the theory arise in the study of plasmas. If a transverse electric wave penetrates a plasma, then its behavior within the plasma is described by the equation

$$\ddot{u}(k,x) + \left[k^2 - \frac{4\pi e^2 \omega^2 N(x)}{m(\omega^2 - \nu^2)}\right] u(k,x) = 0 \quad , \tag{5.1}$$

where $N(x)$ represents the electron density of the medium, ν the collision frequency of the electrons, ω the frequency of the incident wave, e and m the charge and mass of an electron, respectively, and all other symbols are as previously defined. For very high frequencies or in the case of a tenuous plasma, ν^2 is very much less than ω^2 and may be neglected. Thus, Eq. (5.1) becomes

$$\ddot{u}(k,x) + \left[k^2 - \frac{4\pi e^2}{m} N(x)\right] u(k,x) = 0 \quad . \tag{5.2}$$

Equation (5.2) is exactly the same as the Schrödinger equation considered previously with the term $V(x)$ replaced by

$$V(x) = \frac{4\pi e^2}{m} N(x) \quad . \tag{5.3}$$

Thus, the electron density of a plasma may be deduced from the characteristics of the reflections produced, and it may be concluded that the theory presented here is applicable to the investigation of plasmas, including shock waves and re-entry sheaths.

Waves propagating in a magnetoionic medium also obey Eq. (5.2) whenever the electric vector of the wave lies parallel to the direction of the superimposed magnetic field. For this reason, the theory is also applicable to the design of an ionosphere probe. This would be a particularly important application because, using this theory, it would be possible to make a measurement in an interval of time that was short compared to the rate of change of the ionospheric structure; consequently, information concerned with the space-time development of the ionosphere could be obtained.

APPENDIX A

REMOTE MEASUREMENT OF THE INDEX-OF-REFRACTION PROFILE MATHEMATICAL THEORY

I. Introduction

The problem to be considered here is that of determining the index-of-refraction profile along some prescribed path from measurements made external to that path. To make this determination, a sharp impulse of electromagnetic energy will be propagated into a medium whose index of refraction changes from point to point. As the impulse traverses the medium, the energy stored in its electric and magnetic fields will be continuously redistributed, and a portion of the energy will be reflected back toward the source. In the following, it will be shown first, that the characteristics of the reflected impulse are uniquely determined by the index-of-refraction profile, and second, that the index-of-refraction profile can be determined from measurements of the reflection. From a physical point of view, this fact represents a possible solution to the problem because the reflected impulse can be measured at an arbitrary point located external to the path.

Any wave, $w(k,x)$, propagating in a medium with index of refraction $u(x)$ satisfies the Maxwell wave equation

$$\ddot{w}(k,x) + k^2 n^2(x) w(k,x) = 0$$

where k is the wave number or spatial frequency of the wave. For reasons of mathematical expediency, however, we shall at first concern ourselves not with the Maxwell wave equation, but with the Schrödinger wave equation

$$\ddot{u}(k,x) + \left[k^2 - V(x)\right] u(k,x) = 0$$

where $V(x)$, called the scattering potential, is analogous to the index-of-refraction profile. These two equations can be tied together by assuming that $n(x) = 1$ and that $V(x) = 0$ for $x < 0$. Thus, the two equations are identical for $x < 0$ and we can assume that $u(k,x) = w(k,x)$ for $x < 0$.

Subsequently, in Section II of this Appendix, it will be shown that the incident and reflected waves are related by a certain operator S , called the scattering matrix. Study of this operator will yield considerable insight to the nature of the reflection coefficient. Then, in Section III, it will be shown that the scattering potential, $V(x)$, can be determined if the reflection coefficient is known. In Section IV it will be shown how $n(x)$ can be determined from $V(x)$.

II. The Scattering Matrix

Scattering, or radiative transfer, processes are of fundamental importance in many fields of mathematical physics such as electromagnetic theory, quantum mechanics, neutron diffusion, etc. Usually, the aim is to determine the asymptotic scattered field in terms of the incident field and some prescribed potential field or medium which is responsible for the scattering. However, in this study, interest will be focused on the inverse problem, that of specifying the characteristics of the scatterer in terms of the incident and scattered fields.

This discussion will be concerned with the one-dimensional case which is governed by the Schrödinger wave equation

$$\ddot{u}(k,x) + \left[k^2-V(x)\right]u(k,x) = 0 \quad , \tag{A-1}$$

and it will be assumed that

$$V(x) = 0 \quad \text{for} \quad x < 0 \tag{A-2}$$

throughout this discussion, although the results to be obtained will be valid, or can be easily generalized, when a less severe condition replaces Eq. (A-2). For purposes of illustration, it will also be assumed that

$$V(x) = 0 \quad \text{for} \quad x > x_0 \quad . \tag{A-3}$$

However, it will become apparent (because x_0 nowhere enters into the argument) that this condition is invoked purely for purposes of illustration and represents no limitation to the theory.

Let $u(k,x)$ be any solution of Eq. (A-1). Then it can be decomposed into two parts, the incident and the reflected wave:

$$u(k,x) = u_i(k,x) + u_r(k,x) \quad . \tag{A-4}$$

Consider first the incident wave $u_i(k,x)$; it can be further decomposed into two parts, one incident on the perturbation from the left:

$$u_i(k,x) = A_- e^{ikx} \quad \text{for} \quad x < 0 \tag{A-5}$$

and one incident from the right:

$$u_i(k,x) = A_+ e^{-ikx} \quad \text{for} \quad x > x_o \quad . \tag{A-6}$$

Let it be emphasized at this point that the only significance of the condition implied by Eq. (A-3) is that it allows one to speak of the "right side of the perturbation"; if that condition had not been invoked, it would be necessary to write Eq. (A-6) as

$$\lim_{x \to \infty} \left[u(k,x) - A_+ e^{-ikx}\right] = 0 \quad . \tag{A-7}$$

In a similar manner, the reflected wave can be decomposed into two components, a right-hand component and a left-hand component:

$$u_r(k,x) = B_- e^{-ikx} \quad \text{for} \quad x < 0 \tag{A-8}$$

and

$$u_r(k,x) = B_+ e^{ikx} \quad \text{for} \quad x > x_o \quad . \tag{A-9}$$

Combining the above notation,

$$u(k,x) = A_- e^{ikx} + B_- e^{-ikx} \quad \text{for} \quad x < 0 \tag{A-10}$$

and

$$u(k,x) = A_+ e^{-ikx} + B_+ e^{ikx} \quad \text{for} \quad x > x_0 \quad . \tag{A-11}$$

It is convenient to consider the incident wave components, A_- and A_+, as being "prescribed" quantities and the reflected or scattered components, B_- and B_+, as being "measured" quantities. Clearly, $B_\pm$ depends in a linear fashion upon both $A_\pm$ and upon $V(x)$. This dependence can be easily expressed in terms of an operator S, the scattering matrix, a notation introduced by Heisenberg in 1943:

$$\begin{pmatrix} B_- \\ B_+ \end{pmatrix} = S \begin{pmatrix} A_- \\ A_+ \end{pmatrix} \quad . \tag{A-12}$$

This matrix S is a 2 x 2 square matrix having the elements

$$S = \begin{pmatrix} r(k) & \tau(k) \\ t(k) & \rho(k) \end{pmatrix} \quad . \tag{A-13}$$

Clearly, these elements are determined by $V(x)$ alone and are functions of the complex variable k. Before proceeding to the actual discussion of the inverse problem, it is expedient to derive some relations concerning the elements of S.

The first item of interest is the fact that

$$|A_-|^2 + |A_+|^2 = |B_-|^2 + |B_+|^2 \quad . \tag{A-14}$$

Equation (A-14) is no more than a statement of the first law of thermodynamics. The term on the left of Eq. (A-14) is a measure of the energy incident on the scatterer; the term on the right is a measure of the scattered energy; and because $V(x)$ is real, the scattering process is lossless and the equality must hold. From a mathematical point of view, Eq. (A-14) can be proved quite easily by considering the Wronskian of two solutions of Eq. (A-1).

If u and v are two solutions of Eq. (A-1) (or, for that matter, of any differential equation), then $W(u,v)$, the Wronskian of these solutions, is defined as the determinant

$$W(u,v) = \begin{vmatrix} u & v \\ \dot{u} & \dot{v} \end{vmatrix} \tag{A-15}$$

where the dots indicate differentiation with respect to x. It will first be demonstrated that $W(u,v)$ is a constant. Differentiate $W(u,v)$ with respect to x to obtain

$$\frac{d}{dx} W(u,v) = \begin{vmatrix} u & v \\ \ddot{u} & \ddot{v} \end{vmatrix} \quad . \tag{A-16}$$

Substitute the values of $\ddot{u}$ and $\ddot{v}$ into Eq. (A-16):

$$\frac{d}{dx} W(u,v) = -uv \begin{vmatrix} 1 & 1 \\ \left[k^2-V(x)\right] & \left[k^2-V(x)\right] \end{vmatrix} \quad . \tag{A-17}$$

Then

$$\frac{d}{dx} W(u,v) = 0$$

and

$$W(u,v) = \text{constant} \quad . \tag{A-18}$$

Now to prove Eq. (A-14) let u be the solution given by Eqs. (A-10) and (A-11) and let v be the complex conjugate of u . Then,

$$W = 2ik \begin{vmatrix} B_- & A_-^* \\ A_- & B_-^* \end{vmatrix} \quad \text{for} \quad x < 0 \tag{A-19}$$

and

$$W = 2ik \begin{vmatrix} A_+ & B_+^* \\ B_+ & A_+^* \end{vmatrix} \quad \text{for} \quad x > x_0 \quad . \tag{A-20}$$

Because W is constant, these two expressions can be equated to obtain Eq. (A-14).

It will next be shown that S is unitary; this follows immediately from the fact that (a) S transforms the vector $\begin{pmatrix} A_- \\ A_+ \end{pmatrix}$ into the vector $\begin{pmatrix} B_- \\ B_+ \end{pmatrix}$ (see Eq. (A-12)), (b) the length of these two vectors is equal (see Eq. (A-14)), and (c) any operator which transforms a vector, having a finite number of components, into another vector of equal length is unitary. An algebraic proof of this statement can be obtained by substituting values of B_- and B_+ , obtained from Eq. (A-12), into Eq. (A-14) to obtain

$$|A_-|^2\left[1-|r|^2-|t|^2\right] + |A_+|^2\left[1-|\tau|^2-|\rho|^2\right] - A_-A_+^*\left[r\tau^*+t\rho^*\right]$$
$$- A_-^*A_+\left[r^*\tau+t^*\rho\right] = 0 \quad . \tag{A-21}$$

Because A_- and A_+ are arbitrary, it is necessary that

$$|r|^2 + |t|^2 = 1 \quad , \tag{A-22}$$

$$|\rho|^2 + |\tau|^2 = 1 \tag{A-23}$$

and

$$r\tau^* + t\rho^* = 0 \quad . \tag{A-24}$$

By making use of these last three expressions, it follows immediately that

$$S^{-1} = S_T^* \tag{A-25}$$

which proves the assertion.

The last result of interest at this time is the reciprocity law:

$$t = \tau \quad . \tag{A-26}$$

To prove the reciprocity relation, first write down the matrix representation of $u^*(k,x)$ remembering that, in the conjugate wave, the roles of the incident and scattered waves are interchanged:

$$A^* = SB^* \quad . \tag{A-27}$$

Then

$$B^* = S_T^* A^* \quad . \tag{A-28}$$

However, the complex conjugate of Eq. (A-12) is

$$B^* = S^* A^* \quad . \tag{A-29}$$

Comparison of Eq. (A-28) with Eq. (A-29) shows that

$$S = S_T \tag{A-30}$$

which proves the assertion.

As mentioned previously, the vector $\begin{pmatrix} A_- \\ A_+ \end{pmatrix}$ can be considered to be arbitrary. Thus, by selecting $A_- = 0$ and $A_+ = 1$, we obtain

$$u(k,x) = e^{ikx} + r(k)e^{-ikx} \ , \text{ for } \ x < 0 \tag{A-31}$$

and

$$\lim_{x \to \infty} \left[u(k,x) - t(k)e^{ikx} \right] = 0 \quad . \tag{A-32}$$

III. <u>The Inverse Scattering Problem</u>

The wave $u(k,x)$ is assumed to satisfy the Schrödinger wave equation

$$\ddot{u}(k,x) + \left[k^2 - V(x) \right] u(k,x) = 0 \tag{A-33}$$

where the scattering potential, $V(x)$, is a real, piecewise continuous function of x and

$$V(x) = 0 \ \text{ for } \ x < 0 \quad . \tag{A-34}$$

Consequently, $u(k,x)$ can be defined for negative values of x as

$$u(k,x) = e^{ikx} + r(k)e^{-ikx} \ , \ x < 0 \quad . \tag{A-35}$$

The inverse scattering problem is that of constructing an expression for $V(x)$, valid for all x , from the known reflection coefficient, $r(k)$.

Let $v(k,x)$ be a wave solution of the differential equation

$$\ddot{v}(k,x) + k^2 v(k,x) = 0 \tag{A-36}$$

which is equal to $u(k,x)$ for $x < 0$. By the term "wave solution", we mean a solution whose first derivative exists and is continuous for all values of x . Consequently,

$$v(k,x) = e^{ikx} + r(k)e^{-ikx} \text{ , all } x \quad . \tag{A-37}$$

Let $\hat{v}(k,x)$ be another wave solution of Eq. (A-36) and be given by

$$\hat{v}(k,x) = e^{-ikx} - r^*(k)e^{ikx} \tag{A-38}$$

where, as usual, the asterisk indicates a complex conjugate quantity. These two wave solutions were selected because they have the property that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{v(k,x)\hat{v}(k,y)}{|t(k)|^2}\, dk = \delta(x-y) \tag{A-39}$$

where $t(k)$ is the transmission coefficient associated with $r(k)$ and $\delta(x-y)$ is the Dirac delta function,

$$\delta(x-y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-y)}\, dk \quad . \tag{A-40}$$

From another point of view, the function $\dfrac{\hat{v}(k,x)}{|t(k)|^2}$ may be looked upon as the kernel of a certain integral transform acting on $v(k,x)$. Noting the similarity between Eqs. (A-33) and (A-36), we apply the same integral transform to $u(k,x)$ and write the result in terms of a new function $K(x,y)$:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{u(k,x)\hat{v}(k,y)}{|t(k)|^2}\, dk = K(x,y) + \delta(x-y) \quad . \tag{A-41}$$

Now, if this new function $K(x,y)$ does, in fact, exist (we shall prove its existence later), it is easy to show that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{u(j,x)\hat{v}(j,y)v(k,y)}{|t(j)|^2}\,dj\,dk = v(k,x) + \int_{-\infty}^{\infty} K(x,z)v(k,z)dz \qquad \text{(A-42)}$$

whereas

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{u(j,x)\hat{v}(j,y)v(k,y)}{|t(j)|^2}\,dk\,dj = u(k,x) \quad . \qquad \text{(A-43)}$$

Now, under very general conditions (Fubini's theorem) Eqs. (A-42) and (A-43) can be equated to give

$$u(k,x) = v(k,x) + \int_{-\infty}^{\infty} K(x,z)v(k,z)dz \quad . \qquad \text{(A-44)}$$

However, because $u(k,x) = v(k,x)$ for $x < 0$, $K(x,y)$ is triangular in the sense that

$$K(x,y) = 0 \quad \text{for} \quad y < -x \quad . \qquad \text{(A-45)}$$

Thus, Eq. (A-44) becomes

$$u(k,x) = v(k,x) + \int_{-x}^{x} K(x,z)v(k,z)dz \quad . \qquad \text{(A-46)}$$

It is now necessary to prove the existence of $K(x,y)$. To do this, apply the operator $\left[\frac{d^2}{dx^2} - V(x)\right]$ to Eq. (A-46). After substitution from Eqs. (A-33) and (A-36), this gives

$$-k^2u(k,x) = -k^2v(k,x) - V(x)v(k,x) - V(x)\int_{-x}^{x} K(x,y)v(k,y)dy$$

$$+ \frac{d^2}{dx^2}\int_{-x}^{x} K(x,y)v(k,y)dy \quad . \qquad \text{(A-47)}$$

The derivative in Eq. (A-47) is evaluated according to Leibnitz' rule to yield

$$-k^2u(k,x) = -k^2v(k,x) - V(x)v(k,x) + v(k,x)\,\frac{dK(x,x)}{dx}$$

$$+ K(x,x)\,\frac{dv(k,x)}{dx} + v(k,-x)\,\frac{dK(x,-x)}{dx} + K(x,-x)\,\frac{d}{dx}\,v(k,-x)$$

$$+ K_x(x,x)v(k,x) + K_x(x,-x)v(k,-x) + \int_{-x}^{x} K_{xx}(x,y)v(k,y)dy$$

$$- V(x)\int_{-x}^{x} K(x,y)v(k,y)dy \quad . \qquad \text{(A-48)}$$

Next, Eq. (A-46) is multiplied by $-k^2$:

$$-k^2u(k,x) = -k^2v(k,x) + \int_{-x}^{x} K(x,y)\left[-k^2v(k,y)\right] dy \quad . \qquad \text{(A-49)}$$

The bracketed term in the integral of Eq. (A-49) is evaluated from Eq. (A-36). Thus, from Eq. (A-49)

$$-k^2u(k,x) = -k^2v(k,x) + \int_{-x}^{x} K(x,y)\,\frac{d^2}{dy^2}\,v(k,y)dy \quad . \qquad \text{(A-50)}$$

The integration indicated in Eq. (A-50) can be performed twice by parts to yield

$$-k^2u(k,x) = -k^2v(k,x) + K(x,x)\,\frac{dv(k,x)}{dx}$$

$$+ K(x,-x)\,\frac{dv(k,-x)}{dx} - K_y(x,x)v(k,x)$$

$$+ K_y(x,-x)v(k,-x) + \int_{-x}^{x} K_{yy}(x,y)v(k,y)dy \quad . \qquad \text{(A-51)}$$

Next, Eq. (A-51) is subtracted from Eq. (A-48) and use is made of the relation

$$K_x(x,x) + K_y(x,x) = \frac{dK(x,x)}{dx} \qquad \text{(A-52)}$$

$$0 = \int_{-x}^{x} \left\{K_{xx}(x,y)-K_{yy}(x,y)-v(x)K(x,y)\right\} v(k,y)dy$$

$$+ \left\{2\,\frac{dK(x,x)}{dx} - V(x)\right\} v(k,x) + 2\,\frac{dK(x,-x)}{dx}\,v(k,-x) \quad . \qquad \text{(A-53)}$$

Equation (A-53) will be satisfied if the following three conditions are valid:

$$K_{xx}(x,y) - K_{yy}(x,y) - V(x)K(x,y) = 0 \quad , \qquad \text{(A-54)}$$

$$2\,\frac{dK(x,x)}{dx} = V(x) \quad , \qquad \text{(A-55)}$$

and

$$\frac{dK(x,-x)}{dx} = 0 \quad . \qquad \text{(A-56)}$$

Now, we know from the theory of second order differential equations that a solution of Eq. (A-54) which satisfies the boundary conditions, Eqs. (A-55) and (A-56), does exist. Thus, we have proved the existence of $K(x,y)$ and may assume Eq. (A-46) to be valid.

The next step is to find $K(x,y)$. To this end, apply the operator

$$\frac{1}{2\pi}\int_c e^{-iky}\,dy \tag{A-57)*}$$

to Eq. (A-46) (c is a contour, extending from $K = -\infty$ to $K = +\infty$ and lying in the upper half of the complex K-plane above any poles of $r(k)$). This yields

$$0 = R(x+y) + K(x,y) + \int_{-x}^{x} R(y+z)K(x,z)dz \tag{A-58}$$

where

$$R(x+y) = \frac{1}{2\pi}\int_c r(k)e^{-ik(x+y)}dk \quad . \tag{A-59}$$

Equation (A-58) is a Fredholm integral equation of the second kind and can be solved for $K(x,y)$. Thus, Eqs. (A-55), (A-58), and (A-59) represent a solution to the inverse scattering problem.

The above presentation represents only a theoretical solution of the inverse scattering problem. From a practical point of view, the solution is not yet complete. In order to apply the theory so far expounded, it is necessary that $r(k)$ be known for every value of k . However, because $r(k)$ will be determined experimentally, it can at best be known on a bounded real interval or, even more likely, for only a finite set of discrete frequencies. Now, because not every function $r(k)$ can be used to construct a potential $V(x)$, a very real problem does exist. Accordingly, we wish to determine those requirements which, when placed on $r(k)$, are sufficient to insure that a scattering potential, $V(x)$, can be constructed.

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* For instance, this operator applied to the function $f(y)$ would be written $\frac{1}{2\pi}\int_c f(y)e^{-iky}dy$.

First, let us consider $R(x)$. We state, and shall subsequently prove, that the following requirements are sufficient to insure that $V(x)$ can be constructed from Eq. (A-55):

1. $R(x) = 0$ for $x < 0$,
2. $\dot{R}(x)$ exists,
3. $\ddot{R}(x)$ exists,
4. $R(x)$ is continuous, and
5. $\dot{R}(x)$ is continuous.

To prove (1), we note that if $R(x) = 0$ for $x < 0$, then $K(x,y) = 0$ for $x < -y$ and, consequently, $V(x) = 0$ for $x = 0$. The proof of the remaining four conditions requires somewhat more detail.

Our starting point will be the integral equation

$$0 = R(x+y) + K(x,y) + \int_{-x}^{x} R(y+z)K(x,z)dz \quad . \tag{A-60}$$

If $R(x)$ is continuous, then Eq. (A-60) can be written

$$0 = R(x+y) + K(x,y) + \int_{-y}^{x} R(y+z)K(x,z)dz \quad . \tag{A-61}$$

If $\dot{R}(x)$ exists, Eq. (A-61) can be differentiated with respect to x to obtain

$$0 = \dot{R}(x+y) + K_x(x,y) + K(x,x)R(x+y) + \int_{-y}^{x} K_x(x,z)R(y+z)dz \quad . \tag{A-62}$$

If $\ddot{R}(x)$ exists, then Eq. (A-62) can be differentiated with respect to x to obtain

$$0 = \ddot{R}(x+y) + K_{xx}(x,y) + K(x,x)\dot{R}(x+y) + R(x+y)\frac{d}{dx}K(x,x)$$

$$+ K_x(x,x)R(x+y) + \int_{-y}^{x} K_{xx}(x,y)R(y+z)dz \quad . \qquad \text{(A-63)}$$

Next, differentiate Eq. (A-61) with respect to y to obtain

$$0 = \dot{R}(x+y) + K_y(x,y) + K(x,-y)R(0) + \int_{-y}^{x} K(x,z)\dot{R}(y+z)dz \quad . \qquad \text{(A-64)}$$

If $R(x)$ is continuous and if $R(x) = 0$ for $x < 0$, then $R(0) = 0$ and Eq. (A-64) becomes

$$0 = \dot{R}(x+y) + K_y(x,y) + \int_{-y}^{x} K(x,z)R(y+z)dz \quad . \qquad \text{(A-65)}$$

Next, differentiate Eq. (A-65) with respect to y to obtain

$$0 = \ddot{R}(x+y) + K_{yy}(x,y) + K(x,-y)\dot{R}(0) + \int_{-y}^{x} K(x,z)\ddot{R}(y+z)dz \quad . \qquad \text{(A-66)}$$

If $\dot{R}(x)$ is continuous, then $\dot{R}(0) = 0$ and Eq. (A-66) becomes

$$0 = \ddot{R}(x+y) + K_{yy}(x,y) + \int_{-y}^{x} K(x,z)\ddot{R}(y+z)dz \quad . \qquad \text{(A-67)}$$

Next, the integral in Eq. (A-67) is integrated twice by parts and Eq. (A-67) becomes

$$0 = \ddot{R}(x+y) + K_{yy}(x,y) + K(x,x)\dot{R}(x+y) - K_y(x,x)R(x+y)$$

$$+ \int_{-y}^{x} K_{yy}(x,z)R(y+z)dz \quad . \qquad \text{(A-68)}$$

Next, Eq. (A-68) is subtracted from Eq. (A-63) and, from this result, the product of $2 \frac{d}{dx} K(x,x)$ and Eq. (A-61) is subtracted. This yields

$$0 = K_{xx}(x,y) - K_{yy}(x,y) - 2K(x,y) \frac{d}{dx} K(x,x)$$

$$+ \int_{-y}^{x} R(y+z)\left[K_{xx}(x,z)-K_{yy}(x,z)-2K(x,z) \frac{d}{dx} K(x,x)\right]dz \quad . \qquad \text{(A-69)}$$

Obviously, Eq. (A-69) is satisfied if

$$K_{xx}(x,y) - K_{yy}(x,y) - 2K(x,y) \frac{d}{dx} K(x,x) = 0 \quad . \qquad \text{(A-70)}$$

Subtracting Eq. (A-70) from Eq. (A-54), we obtain the desired result:

$$V(x) = 2 \frac{d}{dx} K(x,x) \quad . \qquad \text{(A-71)}$$

Thus, the sufficiency of conditions (1) to (5) has been demonstrated and we see that if $r(k)$ is a function whose transform $B(x)$ satisfies these conditions (as well as the conditions specified in II), then the construction will be successful.

IV. The Determination of $n(x)$

Now that we have determined the scattering potential, $V(z)$, which appears in the equation

$$\ddot{u}(k,z) + \left[k^2-V(z)\right] u(k,z) = 0 \quad , \qquad \text{(A-72)}$$

our next task will be to find the index-of-refraction profile, $n(x)$, which appears in the equation

$$\ddot{u}(k,x) + k^2n^2(x)w(k,x) = 0 \quad . \qquad \text{(A-73)}$$

The requirements imposed on $V(z)$,

$$V(z) = 0 \quad \text{for} \quad z < 0 \tag{A-74}$$

and

$$\lim_{z \to \infty} V(z) = 0 \quad , \tag{A-75}$$

and those which apply to $n(x)$,

$$n(x) = 1 \quad \text{for} \quad x < 0 \tag{A-76}$$

and

$$\lim_{x \to \infty} n(x) = 1 \quad , \tag{A-77}$$

are sufficient to insure that

$$u(k,x) = w(k,x) \quad \text{for} \quad x < 0 \tag{A-78}$$

and

$$\lim_{x \to \infty} \left[u(k,x) - w(k,x) \right] = 0 \quad . \tag{A-79}$$

Thus, Eqs. (A-78) and (A-79) serve to prescribe boundary conditions for Eq. (A-73).

In Eq. (A-72), let $x = x(z)$. Then we obtain

$$\ddot{u}(k,x) + \frac{x''}{(\dot{x})^2} \dot{u}(k,x) + \frac{k^2 - V(z)}{(\dot{x})^2} u(k,x) = 0 \tag{A-80}$$

where the dots indicate differentiation with respect to x and the primes indicate differentiation with respect to z . Next, in Eq. (A-80), let

$$u(k,x) = w(k,x)(\dot{x})^{-\frac{1}{2}} \quad . \tag{A-81}$$

Then we obtain

$$\ddot{w}(k,x) + \left[\frac{k^2}{(\dot{x})^2} + \frac{3(\ddot{x})^2}{4(\dot{x})^2} - \frac{\dddot{x}}{2(\dot{x})^3} - \frac{V(z)}{(\dot{x})^2}\right] w(k,x) = 0 \quad . \tag{A-82}$$

By inspection, we see that Eq. (A-83) is equivalent to Eq. (A-73) if

$$\dot{x} = \frac{1}{n(x)} \tag{A-83}$$

and

$$3\left(\frac{\ddot{x}}{\dot{x}}\right)^2 - 2\left(\frac{\dddot{x}}{\dot{x}}\right) - 4V(x) = 0 \quad . \tag{A-84}$$

Consequently, if we can find an expression for $\dot{x}$ from Eq. (A-84), we can determine $n(x)$ from Eq. (A-83). It is also significant to note that, because $n(x) = 1$ for $x < 1$, $w(k,x)$ in Eq. (A-82) satisfies the prescribed boundary conditions.

In Eq. (A-84), let

$$y = \frac{\ddot{x}}{\dot{x}} \quad . \tag{A-85}$$

Then

$$\dot{y} + y^2 = \frac{\dddot{x}}{\dot{x}} \tag{A-86}$$

and Eq. (A-84) becomes

$$2\dot{y}(z) - y^2(z) + 4V(z) = 0 \quad . \tag{A-87}$$

Equation (A-87) will be recognized as the Ricatti equation. The substitution

$$y(z) = -2\,\frac{\dot{U}(z)}{U(z)} \tag{A-88}$$

transforms it into

$$\ddot{U}(z) - V(z)U(z) = 0 \quad . \tag{A-89}$$

If we solve Eq. (A-89) for $U(z)$, we can then find $\dot{x}$ from

$$\dot{x} = \frac{1}{U^2(z)} \tag{A-90}$$

and $x(z)$ from

$$x(z) = \int \frac{dz}{U^2(z)} \quad . \tag{A-91}$$

Then, inverting $x(z)$ to find $z(x)$, we obtain

$$n(x) = \frac{d}{dx}\, z(x) \quad . \tag{A-92}$$

For $x < 0$, $n(x) = 1$. Thus $z(x) = x$ and $U(z) = 1$ for $z < 0$. By requiring that $U(z)$ and $\dot{U}(z)$ be continuous, we obtain

$$U(0) = 1 \quad \text{and} \quad \dot{U}(0) = 0 \quad . \tag{A-93}$$

These are the boundary conditions which allow us to solve Eq. (A-89) and this completes the problem.

In summary, we are given the function $r(k)$ and want to find $n(x)$. To do this, we must complete the following sequence of operations:

1. Compute $R(x)$ from

$$R(x) = \frac{1}{2\pi} \int_c r(k) e^{-ikx} \, dk \tag{A-94}$$

where c is a contour extending from $k = -\infty$ to $k = +\infty$ and lying in the upper half of the K-plane above the poles of $r(k)$.

2. Find $K(x,y)$ from the integral equation

$$0 = R(x+y) + K(x,y) + \int_{-x}^{x} R(y+z) K(x,z) dz \quad . \tag{A-95}$$

3. Find $V(x)$ from

$$V(x) = 2 \frac{d}{dx} K(x,x) \quad . \tag{A-96}$$

4. Find $U(z)$ from

$$\ddot{U}(z) - V(z) U(z) = 0 \quad ; \tag{A-97}$$

$$U(0) = 1 \text{ and } \dot{U}(0) = 0 \quad . \tag{A-98}$$

5. Find $x(z)$ from

$$x(z) = c \int \frac{dz}{U^2(z)} \quad . \tag{A-99}$$

6. Invert $x(z)$ to obtain $z(x)$.

7. Compute $n(x)$ from

$$n(x) = U^2(x) \quad . \tag{A-100}$$

The constant c in Eq. (A-99) can be evaluated if one value of $n(x)$, i.e., $n(0)$ is known.

APPENDIX B

THE MEASUREMENT OF THE REFLECTION COEFFICIENT, r(k)

The mathematical theory for the remote measurement of the index of refraction, which was advanced earlier in this report, depended succinctly upon a certain quantity, $r(k)$, called the reflection coefficient. In this section, we consider the manner in which this quantity may be measured.

The reflection coefficient, $r(k)$, first appeared in the equation

$$u(k,x) = e^{ikx} + r(k)e^{-ikx} \quad , \tag{B-1}$$

where $u(k,x)$ was assumed to be a solution of the (separated) wave equation

$$\ddot{u}(k,x) + k^2 u(k,x) = 0 \quad . \tag{B-2}$$

Equation (B-2) had, in turn, been obtained from the (total) wave equation

$$\frac{\partial^2}{\partial x^2} u(k,x,t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} u(k,x,t) = 0 \tag{B-3}$$

by separation of the temporal factor $\exp(-ikct)$; that is, by assuming that

$$u(k,x,t) = u(k,x)e^{-ikct} \quad . \tag{B-4}$$

Thus, we can write

$$u(k,x,t) = e^{ik(x-ct)} + r(k)e^{-ik(x+ct)} \quad . \tag{B-5}$$

Now, $u(k,x,t)$ as defined by Eq. (B-5) represents a particular solution of Eq. (B-3); a general solution can be obtained by multiplying by $f(k)$, an arbitrary function of the wave number k only. Let

$$g(k) = r(k)f(k) \quad , \tag{B-6}$$

and this general solution is

$$u_g(k,x,t) = f(k)e^{ik(x-ct)} + g(k)e^{-ik(x+ct)} \quad . \tag{B-7}$$

This general solution represents the sum of two waves, a forward (toward $x = +\infty$) traveling wave, $f(k)\exp[ik(x-ct)]$, and a backward (toward $x = -\infty$) traveling wave, $g(k)\exp[-ik(x+ct)]$. Both waves travel with a velocity c

$$c = \frac{1}{(u_o\varepsilon_o)^{\frac{1}{2}}} \quad . \tag{B-8}$$

By further inspection, we see that the forward traveling wave is actually the sum of an infinite number of sinusoidal components, each component having a spatial frequency, or wave number, k, and an amplitude of $f(k)$. A similar interpretation can be given to the backward traveling wave. Accordingly, the total waves can be obtained by summation:

$$F(x-ct) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(k)e^{ik(x-ct)}\,dk \tag{B-9}$$

and

$$G(x+ct) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k)e^{-ik(x-ct)}\,dk \quad . \tag{B-10}$$

Now, the two functions $F(x-ct)$ and $G(x+ct)$ are the space-time profiles of the incident and reflected waves, respectively. They represent two measurable quantities, and it is these two quantities which we shall measure.

Now notice that $F(x-ct)$ and $G(x+ct)$ are Fourier transforms of the functions $f(k)$ and $g(k)$, respectively. Accordingly, once $F(x-ct)$ and $G(x+ct)$ have been measured, $f(k)$ and $g(k)$ can be obtained by inversion:

$$f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x-ct) e^{-ik(x-ct)} \, d(x-ct) \tag{B-11}$$

and

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(x+ct) e^{ik(x+ct)} \, d(x+ct) \quad . \tag{B-12}$$

Then, $r(k)$ may be found from

$$r(k) = \frac{g(k)}{f(k)} \quad . \tag{B-13}$$

In practice, a very short impulse, $F(x-ct)$, will be transmitted into the region of nonconstant index of refraction and a, presumably longer, impulse $G(x+ct)$ will be reflected. The requirement that $F(x-ct)$ be a very short impulse stems from the fact that the index-of-refraction profile will be slowly changing with time. If $F(x-ct)$ is too long, or, in other words, if too much time is taken to make the measurement, then a "smearing" will take place which will invalidate the results. From another point of view, we wish to measure the value of the reflection coefficient over as broad a band of spatial frequencies, k, as is practicable. To do this, it is necessary that the spectrum of $F(x-ct)$ be as broad as possible and one way of obtaining a broad spectrum is to use a very short impulse.

APPENDIX C

AN ALTERNATE DERIVATION OF THE INVERSE SCATTERING THEORY

In this section we shall present an alternate derivation of the inverse scattering theory and obtain, as a consequence, a different (but equivalent) inversion formula which may be useful under some circumstances. We shall also present examples to demonstrate the utility of both inversion formulae.

Formally, we are given the differential equation

$$\ddot{u}(k,x) + \left[k^2 - V(x)\right] u(k,x) = 0 \tag{C-1}$$

where

$$V(x) = 0 \quad \text{for} \quad x < 0 \tag{C-2}$$

and $u(k,x)$ is known for $x \le 0$. We wish to find the value of $V(x)$ for $x \ge 0$. Now, assume that

$$u(k,x) = f(k)e^{ikx} + g(k)e^{-ikx} \ , \ x \le 0 \quad . \tag{C-3}$$

Following the same procedure as in Appendix A, we define the two unperturbed solutions

$$v(k,x) = f(k)e^{ikx} + g(k)e^{-ikx} \tag{C-4}$$

and

$$\hat{v}(k,x) = f^*(k)e^{-ikx} - g^*(k)e^{ikx} \quad . \tag{C-5}$$

Next, we define the function $|h(k)|^2$ by

$$|h(k)|^2 = |f(k)|^2 - |g(k)|^2 \quad . \tag{C-6}$$

Consequently, $v(k,x)$ has been normalized so that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{v(k,x)\hat{v}(k,y)}{|h(k)|^2}\,dk = \delta(x-y) \quad . \tag{C-7}$$

Now, assume that there exists a function $K(x,y)$ such that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{u(k,x)\hat{v}(k,y)}{|h(k)|^2}\,dk = \delta(x-y) + K(x,y) \quad . \tag{C-8}$$

The existence of this function, $K(x,y)$, may be verified by exactly the same method used in Appendix A, Eqs. (A-47) to (A-56), and we find that it must satisfy the equations

$$K_{xx}(x,y) - K_{yy}(x,y) - V(x)K(x,y) = 0 \quad , \tag{C-9}$$

$$\frac{d}{dx}K(x,x) = \frac{1}{2}V(x) \tag{C-10}$$

and

$$K(x,-x) = \text{constant} \quad . \tag{C-11}$$

By the same token, using the methods of Appendix A, Eqs. (A-42) to (A-46), we see that

$$u(k,x) = v(k,x) + \int_{-x}^{x} K(x,z)v(k,z)\,dz \quad . \tag{C-12}$$

Now, apply the operator

$$\frac{1}{2\pi}\int_C e^{-iky}\,dk \tag{C-13}$$

to Eq. (C-12) to obtain

$$0 = G(y+x) + \int_{-x}^{x} K(x,z)F(y-z)dz + \int_{-x}^{x} K(x,z)G(y+z)dz \quad , \tag{C-14}$$

where

$$F(x) = \frac{1}{2\pi}\int_C f(k)e^{-ikx}\,dk \tag{C-15}$$

and

$$G(x) = \frac{1}{2\pi}\int_C g(k)e^{-ikx}\,dk \quad . \tag{C-16}$$

Equation (C-14) is the alternate inversion formula.

To indicate the equivalence of the two inversion formulae, as well as to demonstrate one method of solution, we shall consider two examples. In the first example, let

$$f(k) = \frac{i(k-i)}{(k+i)^2} \tag{C-17}$$

and

$$g(k) = \frac{-i}{(k+i)^3} \quad . \tag{C-18}$$

Then

$$r(k) = \frac{g(k)}{f(k)} = \frac{-1}{k^2+1} \quad . \tag{C-19}$$

Using the calculus of residues, we may easily ascertain that

$$F(x) = (1-2x)e^{-x} \;,\; x \geq 0$$
$$F(x) = 0 \qquad ,\; x < 0 \quad , \tag{C-20}$$

$$G(x) = \tfrac{1}{2} x^2 e^{-x} \;,\; x \geq 0$$
$$G(x) = 0 \qquad ,\; x \leq 0 \quad , \tag{C-21}$$

and that

$$R(x) = \sinh x \;,\; x \geq 0$$
$$R(x) = 0 \qquad ,\; x \leq 0 \quad . \tag{C-22}$$

Consequently, we are concerned with the two integral equations

$$0 = G(x+y) + \int_{-x}^{y} K(x,z)F(y-z)dz + \int_{-y}^{x} K(x,z)G(y+z)dz \tag{C-23}$$

and

$$0 = R(x+y) + K(x,y) + \int_{-y}^{x} K(x,z)R(y+z)dz \quad . \tag{C-24}$$

We shall solve Eq. (C-24) first.

Note that

$$P(p)R(y) = 0 \tag{C-25}$$

where

$$P(p) = p^2-1 \tag{C-26}$$

and

$$p = \frac{d}{dy} \quad . \tag{C-27}$$

Applying this operator to Eq. (C-24), we obtain

$$0 = P(p)R(x+y) + P(p)K(x,y) + \int_{-y}^{x} K(x,z)P(p)R(y+z)dz$$

$$+ K(x,-y)\left[R(0)+\dot{R}(0)\right] - K_y(x,-y)R(0) \quad . \tag{C-28}$$

Now, because $R(0) = 0$, and $\dot{R}(0) = 1$, Eq. (C-28) becomes

$$0 = K_{yy}(x,y) - K(x,y) + K(x,-y) \quad . \tag{C-29}$$

The solution of Eq. (C-29) is

$$K(x,y) = a(x) + b(x) \sinh \sqrt{2}y \tag{C-30}$$

where $a(x)$ and $b(x)$ are arbitrary functions of x alone. By substituting the expression for $K(x,y)$ from Eq. (C-30) into Eq. (C-24) and preforming the indicated integrations, we obtain

$$\begin{aligned} K(x,y) &= -\frac{1}{\sqrt{2}} \frac{\sinh \sqrt{2}x + \sinh \sqrt{2}y}{\cosh \sqrt{2}x} , \quad y \geq -x \\ K(x,y) &= 0 \qquad\qquad\qquad\qquad\quad , \quad y < -x \quad . \end{aligned} \tag{C-31}$$

Consequently, from Eq.(C-10) we easily obtain

$$\begin{aligned} V(x) &= -4 \sinh^2 \sqrt{2}x , \quad x \geq 0 \\ V(x) &= 0 \qquad\qquad\quad , \quad x < 0 \quad . \end{aligned} \tag{C-32}$$

Next, we turn our attention to Eq. (C-23). We obtain easily

$$P(p)F(y) = 0 \tag{C-33}$$

where

$$P(p) = p^2 + 2p + 1 \tag{C-34}$$

and that

$$(p+1)P(p)G(y) = 0 \quad . \tag{C-35}$$

Thus, by applying the operator $(p+1)^3$ to Eq. (C-23), and using the values

$$A(0) = 1 \ , \ \dot{A}(0) = -3 \ , \ \ddot{A}(0) = 5 \tag{C-36}$$

and

$$B(0) = 0 \ , \ \dot{B}(0) = 0 \ , \ \ddot{B}(0) = 1 \quad , \tag{C-37}$$

we obtain

$$0 = K_{yy}(x,y) - K(x,y) + K(x,-y) \quad . \tag{C-38}$$

Equation (C-38) is identical to Eq. (C-29). Consequently, its solution is given by Eq. (C-31) and $V(x)$ by Eq. (C-32).

It is important to note from this example that, while $R(x)$ and $G(x)$ are continuous, $A(x)$ has a discontinuity at the origin. From this, we may assume that it is not necessary that $A(x)$ be continuous. However, it would be incorrect to assume, in this respect, that Eq. (C-23) represents a generalization of Eq. (C-24) as the next example will show.

For the second example, let

$$f(k) = \frac{i}{k+i} \ , \ g(k) = \frac{1}{(k+i)^2} \qquad \text{(C-39)}$$

and

$$r(k) = \frac{g(k)}{f(k)} = \frac{-i}{k+i} \quad . \qquad \text{(C-40)}$$

Then

$$\begin{aligned} R(x) &= -e^{-x} \ , \ x \geq 0 \\ &= 0 \qquad , \ x < 0 \quad . \end{aligned} \qquad \text{(C-41)}$$

In this case, $R(x)$ is discontinuous and we would expect the theory to fail (which it does). $R(x)$ satisfies

$$(p+1)R(x) = 0 \quad . \qquad \text{(C-42)}$$

Thus, by inserting the value of $R(x)$ into Eq. (C-24) and performing the operation $(p+1)$, we obtain

$$0 = K_y(x,y) + K(x,y) - K(x,-y) \quad . \qquad \text{(C-43)}$$

The solution of Eq. (C-43) is

$$K(x,y) = K(x) \quad . \qquad \text{(C-44)}$$

But, from the relation

$$K(x,-x) = \text{constant} \qquad \text{(C-45)}$$

we obtain

$$K(x,x) = \text{constant} \tag{C-46}$$

and

$$V(x) \equiv 0 \text{ , all } x \quad . \tag{C-47}$$

However, the reflection coefficient $r(k) = 0$ corresponds to $V(x) = 0$, not the one given in Eq. (C-40).

Continuing the example, we have

$$\begin{aligned} F(x) &= e^{-x} \text{ , } x \geq 0 \\ F(x) &= 0 \quad \text{ , } x < 0 \end{aligned} \tag{C-48}$$

and

$$\begin{aligned} G(x) &= -xe^{-x} \text{ , } x \geq 0 \\ G(x) &= 0 \qquad \text{ , } x \leq 0 \quad . \end{aligned} \tag{C-49}$$

These functions satisfy the equation

$$(p+1)F(x) = (p+1)^2 G(x) = 0 \quad . \tag{C-50}$$

Thus, by substituting the value of $F(x)$ and $G(x)$ into Eq. (C-23) and performing the operation $(p+1)^2$ we obtain

$$0 = K_y(x,y) + K(x,y) - K(x,-y) \tag{C-51}$$

and, as before, the theory fails.

In concluding this Appendix, let us point out that the second example, used here to demonstrate the failure of the theory, is exactly the same as the example used in Appendix C of the Fourteenth Interim Technical Report (10 July 1962) to demonstrate the validity of the theory. The answer to this unseemly contradiction is that all the work appearing in Appendix C of the Fourteenth Interim Technical Report is incorrect.

APPENDIX D

CALIBRATED TRANSMISSION PATH

The calibrated transmission path will consist of a section of K_u band waveguide (either RG-91/U brass guide or RG-107/U silver guide) approximately one meter long. This waveguide will be carefully assembled and calibrated and will provide the experimenter with a transmission path whose index-of-refraction profile and other pertinent characteristics are accurately known. Initially, experiments will be performed to indicate how variations in the index-of-refraction profile along the line will affect length measurements made using a phase-null or interferometer technique. Later experiments will be aimed at determining how accurately the reflections caused by the variations of the index-of-refraction profile can be measured and how these measurements can be used to more accurately determine the length of the path.

The length of the guide will be determined using the equipment shown in Fig. 6. This equipment will consist of the following components:

1. Huggins Model 615 Oscillator (10-20 Gc., 0.5 mw.),
2. Huggins Model 322 TWT Amplifier (30 db, 1 w.),
3. Waveguide Tee (Demornay Bonardi DBF620),
4. Crossguide Coupler (DBF631),
5. Matched Load (DBF450),
6. Elbow (DBF224),
7. Attenuator, 0-50 db (DBF420),
8. Thermistor Mount (Hewlett-Packard P487B), and
9. Power Meter (Hewlett-Packard 430 C).

To determine the waveguide length using the phase-null system, a short circuit will be placed at one end of the guide and a microwave signal from the phase-null equipment, Fig. 6, fed into the other. The electric field

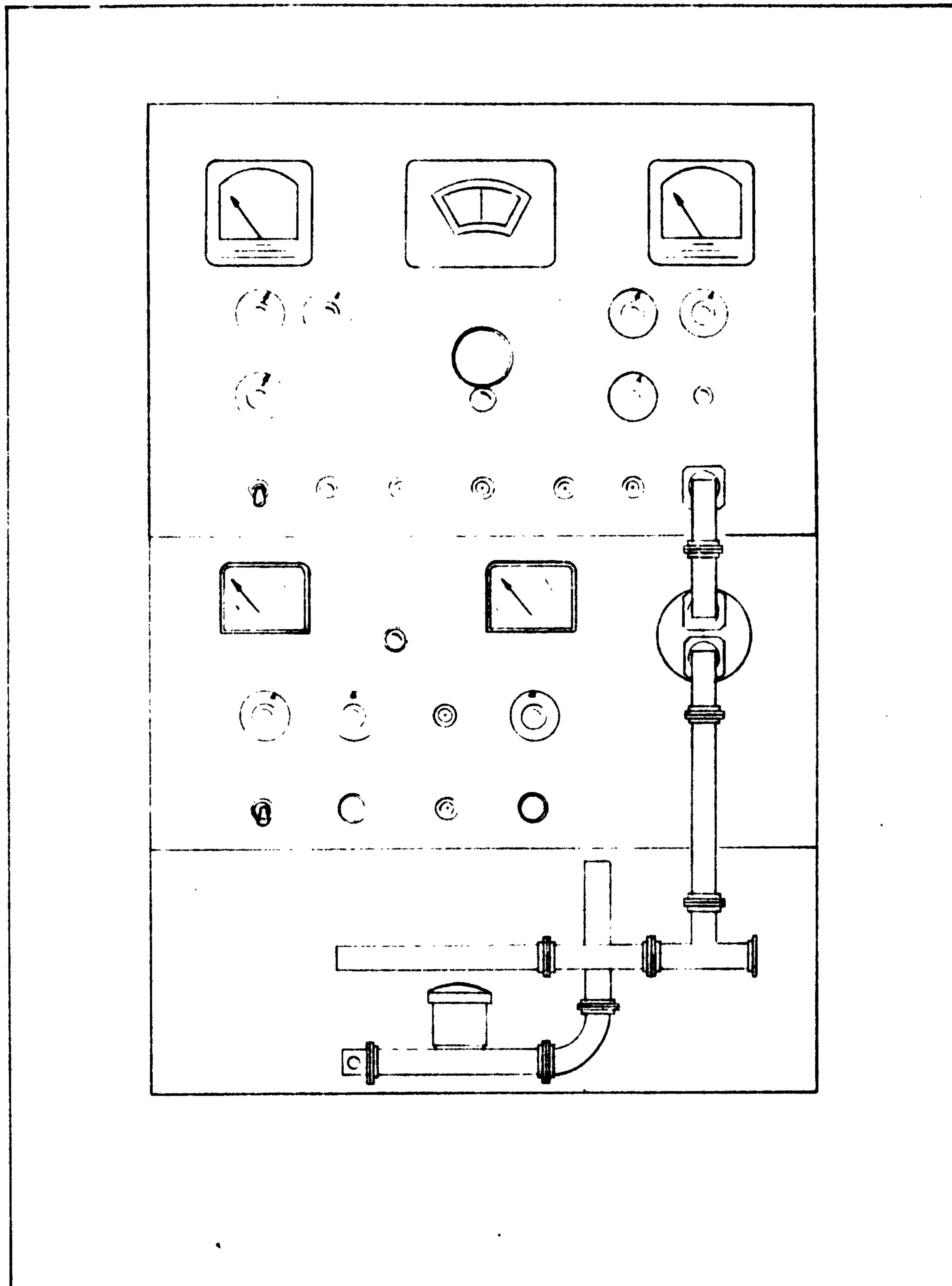

Fig. 6 - Phase-Null Equipment

within the guide will be zero at the short circuited end and, because of the standing waves caused by the short, will also be zero every one-half guide wavelength along the line measured from the short.

The oscillator frequency will be varied until the voltage at the sending end is zero, and the half guide wavelength, λ_1 , corresponding to this frequency will be measured. Then, the length of the guide, L , can be expressed as

$$L = m\lambda_1$$

where m is an integer yet to be determined. Next, increase the oscillator frequency. The voltage at the sending end will increase (as the frequency increases), reach a maximum, and then decrease to zero. Denote the half guide wavelength at which this zero occurs as λ_2 . The waveguide length can now be expressed in terms of λ_2 as

$$L = (m+1)\,\lambda_2\ .$$

From these two results, it follows that

$$L = \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}$$

and, since both λ_1 and λ_2 are known, L can be determined.

Unfortunately, computational difficulties arise at this point because of the necessity of dividing by the difference of two numbers, λ_1 and λ_2 , which are nearly equal. To illustrate, assume that $\lambda_1 = 2$ cm. and that $L = 100$ cm. Then $m = 50$ and

$$\lambda_2 = \frac{100}{51} = 1.960784\ \ldots\ \text{cm.}$$

Now, assume that λ can only be measured to an accuracy of ± 0.001 cm. Then the two measured quantities would be

$$\lambda_1 = 2.000 \text{ cm.},$$

$$\lambda_2 = 1.961 \text{ cm.}$$

and

$$L = \frac{(2.000)(1.961)}{2.000-9.161} = 100.564 \; \ldots .,$$

resulting in an error of 0.564 cm. in L .

To remedy this problem, assume that, as before, λ_1 represents the half guide wavelength at which a zero of the input voltage occurs. Then increase the oscillator frequency noting that the input voltage varies through successive maxima and zeros. Let λ_n be the half guide wavelength at which the n^{th} zero occurs. Then

$$l = m\lambda_1 = (m+n)\lambda_n$$

and

$$l = n\,\frac{\lambda_1\lambda_n}{\lambda_1-\lambda_n} \quad .$$

Now, assume that $\lambda_1 = 2$ cm., $L = 100$ cm., and $n = 40$. Then

$$\lambda_{40} = 1.111 \; \ldots \text{ cm.}$$

Assume, once again, that λ_n can be measured to an accuracy of ± 0.001 cm. Then the measured quantities will be

$$\lambda_1 = 2.000$$

$$\lambda_{40} = 1.111 \quad ,$$

and

$$l = 40 \frac{(2.000)(1.111)}{2.000-1.111} = 99.978$$

or, an error of only 0.022 cm. in L .

These numerical examples serve only to illustrate the nature of the error which will result and also to indicate that the error can be minimized by maximizing the term "n" in the formula

$$L = n \frac{\lambda_1 \lambda_n}{\lambda_1 - \lambda_n} \quad .$$

For a line one meter long and a range of frequencies 12-18 Gc.,the maximum value of n will be approximately 50 (when $\lambda_1 = 2\lambda_{50} = 2$ cm.). To obtain an approximation of the error which can be expected, use the above values to write

$$\Delta L = 50 \frac{(2+\epsilon_1)(1+\epsilon_{50})}{(2+\epsilon_1)-(1+\epsilon_{50})} - 100$$

where ΔL is the error in L and ϵ_1 and ϵ_{50} are the errors in λ_1 and λ_{50} . Expanding the above equations,

$$\Delta L = 50 \frac{\epsilon_1 \epsilon_{50} - \epsilon_1}{1+\epsilon_1-\epsilon_{50}} \quad .$$

Next, assume that $|\epsilon_1| = |\epsilon_{50}| = 0.001$ cm. and maximize the numerator and minimize the denominator to obtain

$$\Delta L < 50 \frac{0.001001}{0.998} = 0.05 \text{ cm.}$$

In other words, with the equipment specified, the length of a one-meter section of guide can be determined to better than ± 1 mm.

Before making measurements, the test equipment, Fig. 6, must be calibrated. To do this, connect a calibrated-movable-short such as Demornay Bonardi DBF969-1 to the test equipment (the open arm of the tee). Then, follow the procedure below:

1. Set the oscillator to that frequency near 12 Gc. where zero power is indicated (on the 430 C power meter).

2. Move the short in a direction away from the tee, noting that the power indicated on the 430 C will increase and then decrease again to zero. Record the distance which the short has moved as λ_1 and return the short to its original position.

3. Increase the oscillator frequency to about 18 Gc. noting that as the frequency is increased, the indicated power varies through successive maxima and zeros. Set the oscillator to that frequency where a zero is obtained and record the total number of zeros as n .

4. Measure this new wavelength by the procedure in (2) and record this value as λ_n .

5. Compute the effective length of the calibration equipment, L_e , from

$$L_e = n \frac{\lambda_1 \lambda_n}{\lambda_1 - \lambda_n} \quad .$$

This length must be subtracted from all subsequent measurements. It should, of course, be measured several times and the results averaged to obtain maximum accuracy.

To measure the length of a section of waveguide, connect the calibrated length measuring equipment to one end of the guide and the short to the other. Repeat the procedure above to determine the combined length of the guide and effective length of the equipment, $L+L_e$. Subtract L_e from the result and the remainder will be the length of the guide.

Once the measuring equipment and test line have been set up and calibrated, the next step will be to introduce artificial dielectric perturbations into the test line and the experimenter will attempt to (a) measure the reflections produced by the perturbations, and (b) use these measurements to prove the validity of the inverse scattering theory.

Whenever a dielectric is introduced into a waveguide, the effect will be to increase the length of the guide. This fact can be demonstrated with a simple example.

Assume that the guide is 100 cm. long and that the operating frequency is 15 Gc. Then, for an air filled guide ($\epsilon_r = 1$), the guide wavelength will be about 2.5 cm. (refer to Fig. 7 which shows the relationship between guide wavelength and frequency for guides filled with various dielectrics). Thus, the length of the guide will be $100 \div 2.5 = 40$ wavelengths. Now, suppose that a block of dielectric having a relative dielectric constant of $\epsilon_r = 20$ and a length of 10 cm. is inserted into the guide. Referring to Fig. 7, one sees that the wavelength in this dielectric is about 0.45 cm. Thus, the line length will be $90 \div 2.5 + 10 \div 0.45 = 145$ wavelengths, an increase of over 100 wavelengths. If one were to attempt to measure the length of this line using the phase-null method just described, the result would be $145 \times 2.5 = 362.5$ cm. It is toward the correction of errors of this nature that the present research is directed.

Initially, the reflection coefficient of the artificial dielectric perturbations will be measured using standard techniques and the data so obtained will be used to prove the validity (and utility) of the inverse scattering theory. However, it is apparent that, in order to measure reflection coefficients of -120 db or less, new techniques will have to be perfected.

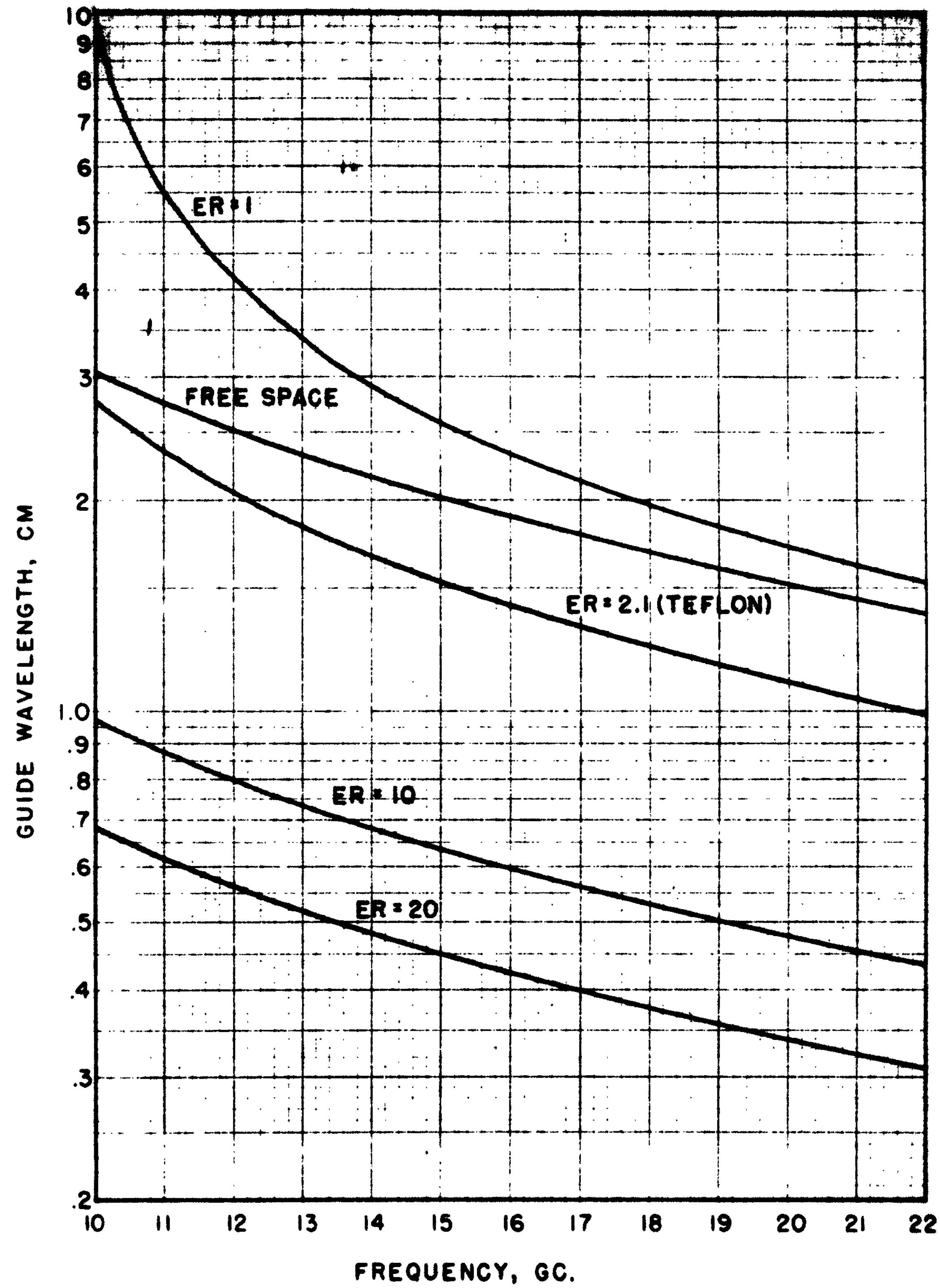

Fig. 7 - Guide Wavelength versus Frequency

APPENDIX E

RADIO FREQUENCY ENERGY SOURCE

For actual index-of-refraction profile measurements, the tunable, C-W source used in the calibration of the transmission path will be replaced by a pulsed source. This pulsed source will consist of a pulse-producing system and a pulse-shaping system. The pulse-producing system will comprise an ordinary K_u-band magnetron and associated modulation circuitry for producing pulses of rf energy (either on a single-shot or a repetitive basis) which are of the order of one microsecond in duration. The pulse-shaping system, at the output of the magnetron, will convert these microsecond-pulses into the extremely short-duration pulses (of the order of a few nanoseconds) that will be necessary for the experiments. Since the development of the pulse-shaping system represents a major area of research effort, very little can be said about the final form of this portion of the pulsed source. However, the design of the pulse-producing portion of the pulsed source (i.e., the modulator and magnetron) can be specified within the existing state of the art.

Experience has shown that a QK319, K_u-band magnetron can be successfully pulsed with an APS-2, hard-tube radar modulator. Since both the QK319 magnetron and the APS-2 modulator are readily available, at very reasonable cost, a pulse-producing system based on these items is contemplated. Some construction work will be necessary in the preparation of a modulator control system, and an appropriate means for monitoring the operation of the pulse-producing system will have to be provided.

A block diagram, showing the essential features of the pulse-producing system, is presented in Fig. 8. Typical features of the APS-2 radar modulator are shown in the schematic diagram of Fig. 9. A triggering signal, provided through contact F on J805 is impressed on the pulse forming network, which is shown in the lower right-hand corner of the diagram. The actual arrangement of the pulse forming network is determined by two relays, K803 and K804, which are controlled, through contacts A, E, and H of J805, from the "pulse duration" selector in the modulator control system. In turn, the arrangement of the pulse forming network determines the duration of the pulses produced by the driver tube, V802, and consequently the duration of the pulses developed by the output tube, V805. Pulse duration of 0.5, 1.0, and 2.0 μsec. is available in all models of the APS-2 modulator.

Normally, the output pulses developed by V805 are delivered directly to the cathode of a 2J22 (S-band) magnetron of a 725-A (X-band) magnetron which is mounted in the lower portion of the APS-2 modulator. However, by means of

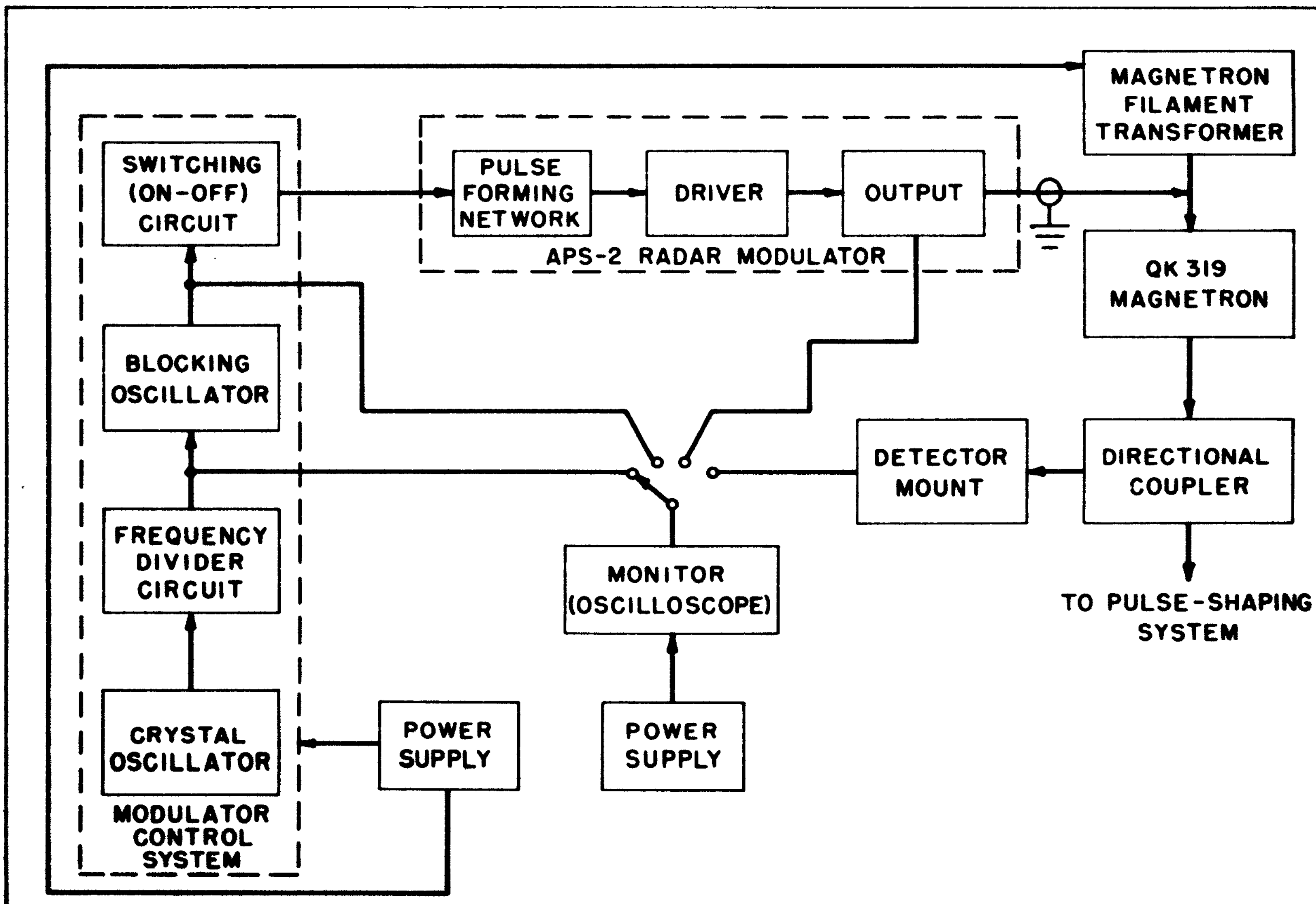

Fig. 8 - Pulse Producing System Block Diagram

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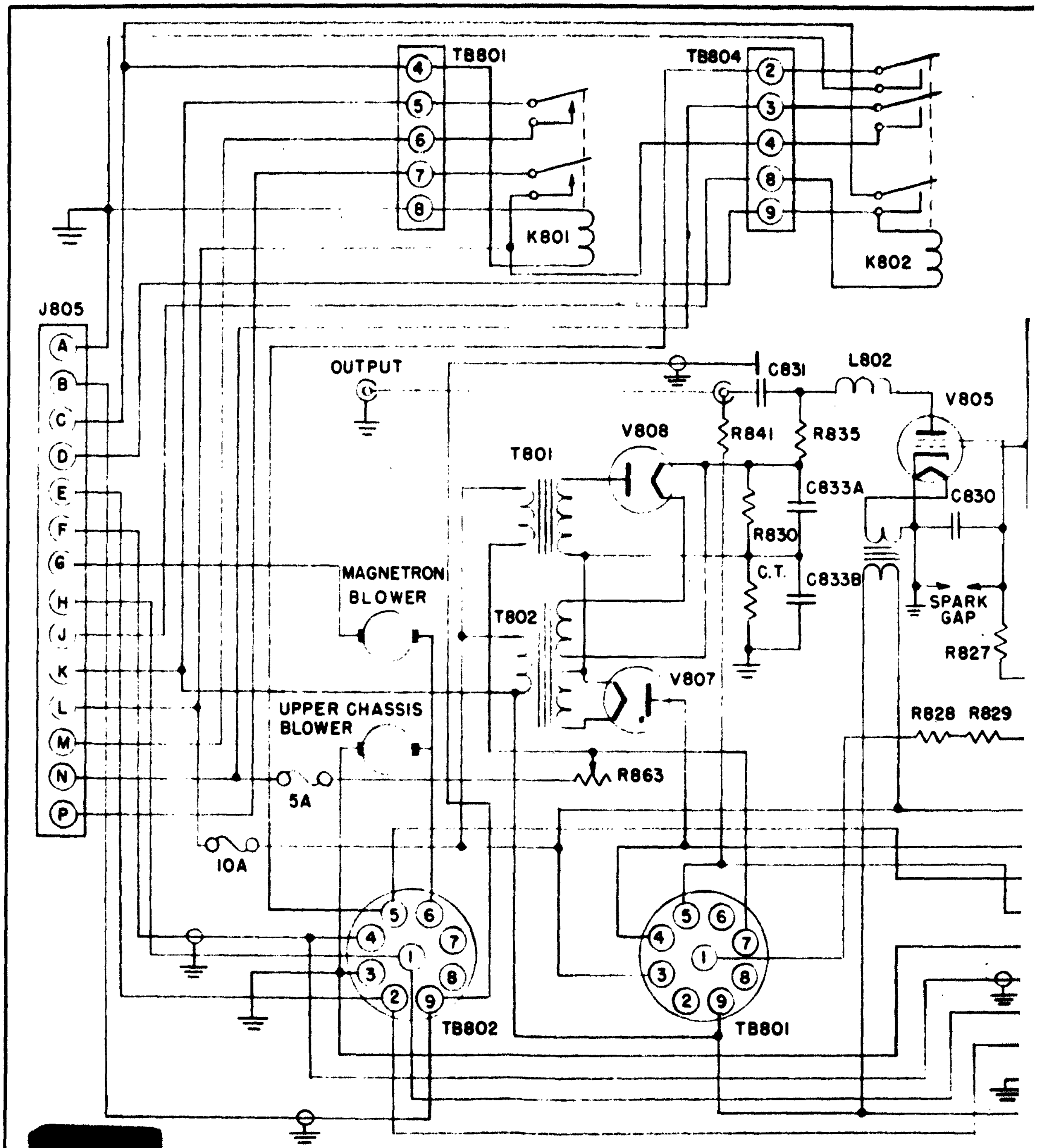

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Fig. 9 - Typical Modulatc

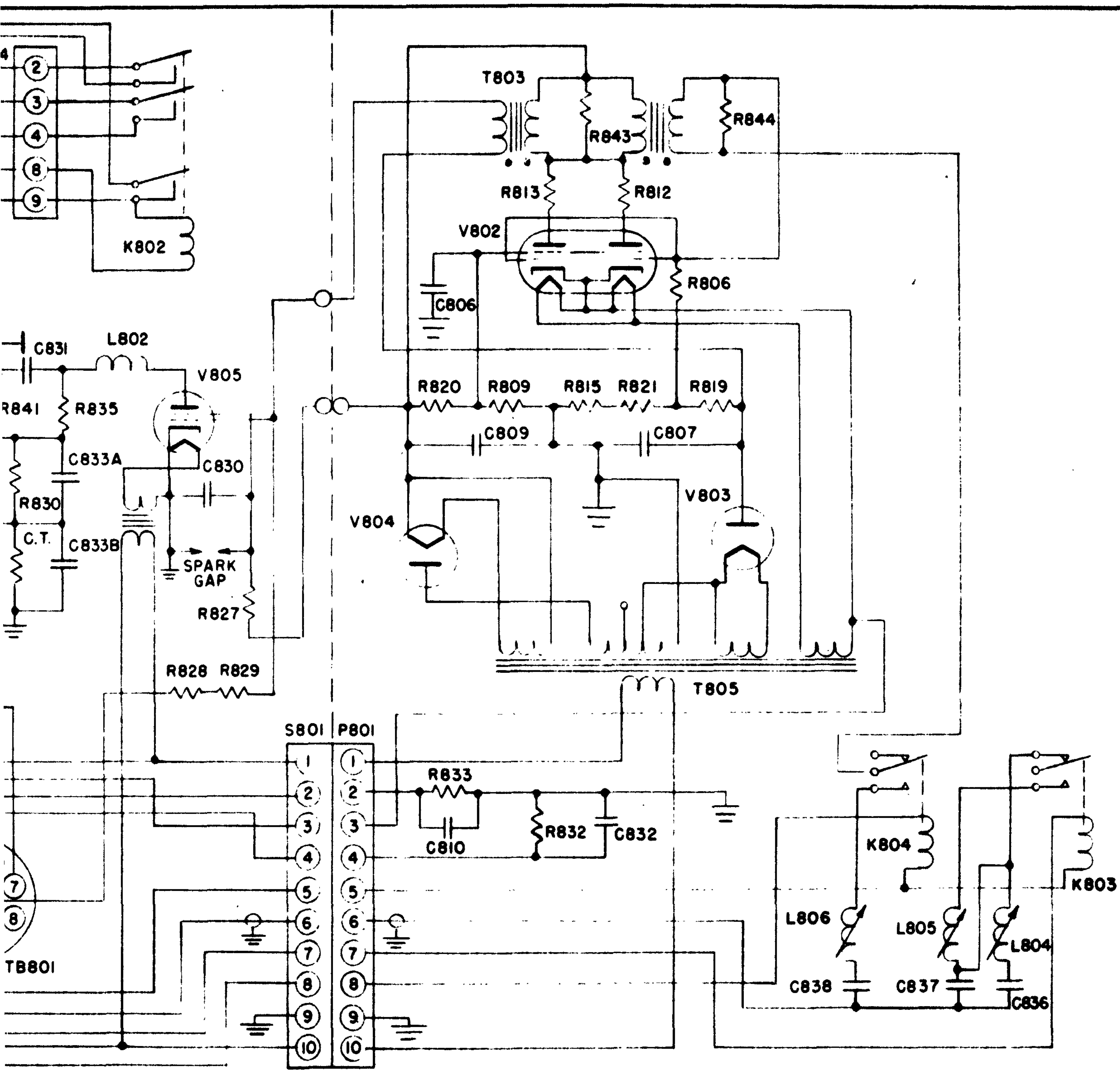

Fig. 9 - Typical Modulator Circuit (APS-2)

a slight modification of the APS-2 modulator circuitry, the output pulses can be delivered to the cathode of a QK319 (K_u-band) magnetron through a short length of high voltage cable. Experience has shown this arrangement to be quite satisfactory in that it provides for a great deal of flexibility in the physical location of the magnetron. In other words, the magnetron, along with its filament transformer, can be mounted at any desired location in an experimental setup (or even moved from one location to another) without the necessity of making complicated changes in waveguide connections to the heavy, rather immobile, modulator unit.

It should be noted that under typical operating conditions, the QK319 magnetron requires a 15-kv.pulse for satisfactory operation. Since the APS-2 radar modulator is capable of producing only a 12-kv.pulse, a modification of the QK319 magnetic circuit is necessary. Although such a modification results in a reduction of the power output of the QK319, peak powers of as much as 15-20 kw. can still be obtained with this magnetron-modulator combination.

Control and triggering of the APS-2 radar modulator can be accomplished with the special modulator control system shown schematically in Fig. 10. This modulator control circuit includes a 10 kc, crystal controlled oscillator which provides a stable reference signal for pulse repetition frequency. Five flip-flop circuits are used as frequency dividers; a suitable switching circuit allows nine different arrangements of the five flip-flop circuits and consequently provides nine different (crystal controlled) pulse repetition frequencies between approximately 250 pps and 1,650 pps. The output of the frequency divider circuit drives a blocking oscillator which provides the trigger pulses for the APS-2 radar modulator through contact F on P805. "On-off" control of the modulator plate supply is accomplished with the circuitry shown in the upper, right-hand portion of Fig. 10. The various voltages required for the modulator control system are furnished by the power supply shown schematically in Fig. 11.

A Tektronix RM16 oscilloscope provides means for monitoring signals within the pulse producing system. For convenience and flexibility of operation, the monitor oscilloscope, APS-2 radar modulator and modulator control system (including its power supply) can be arranged in a portable modulator unit as shown in Fig. 12.

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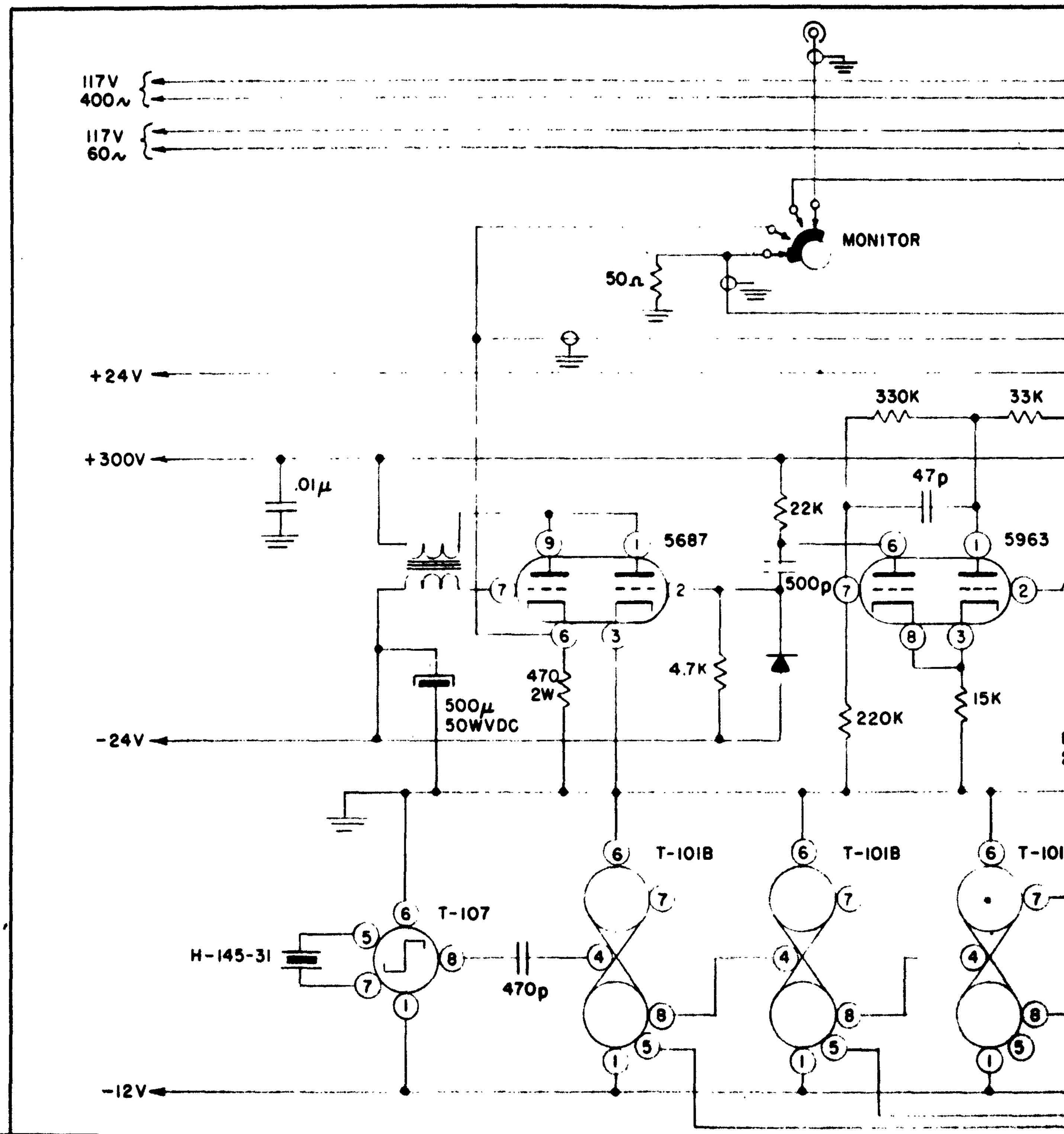

Fig. 10 - Modulator Control

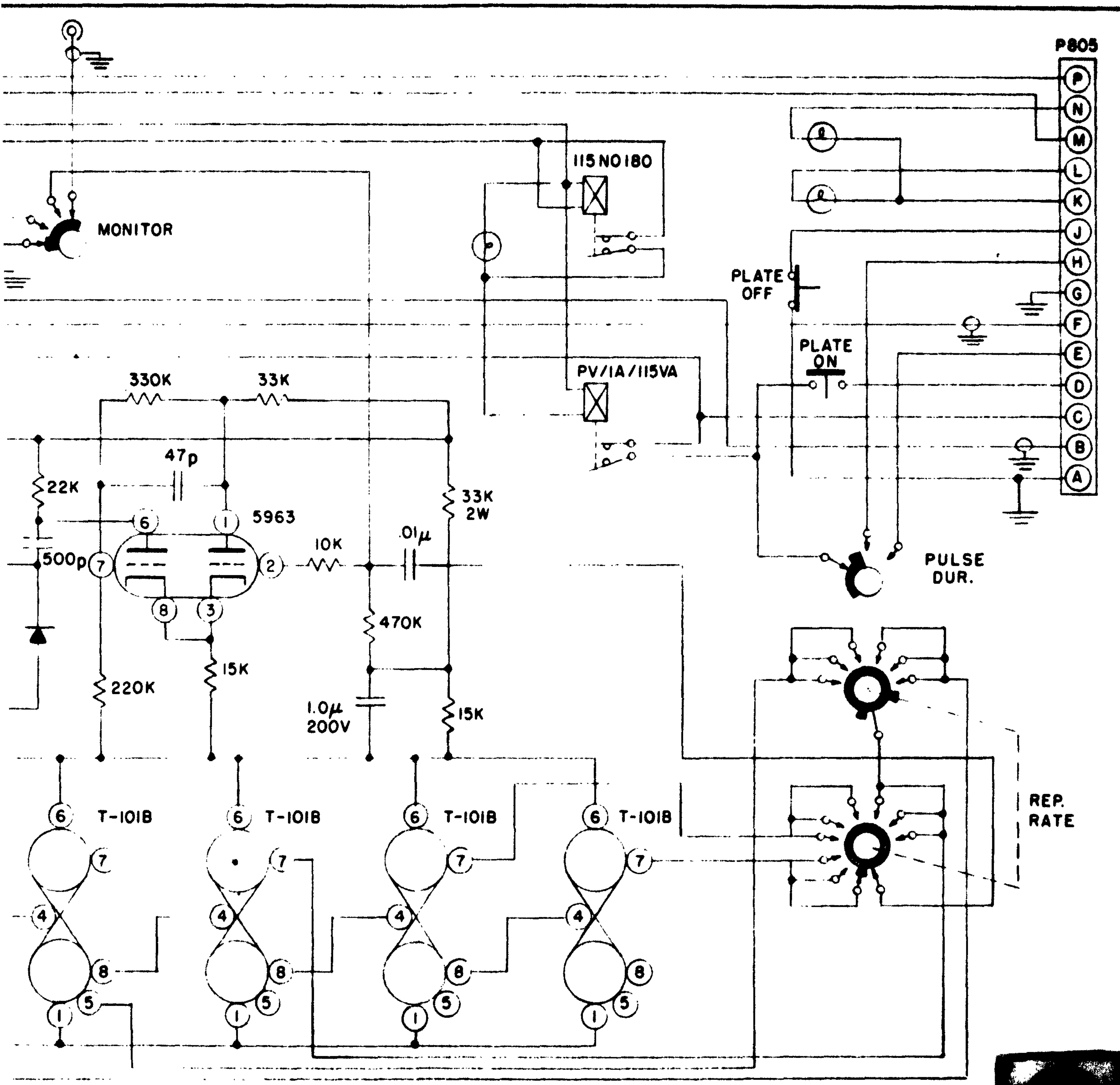

Fig. 10 - Modulator Control System

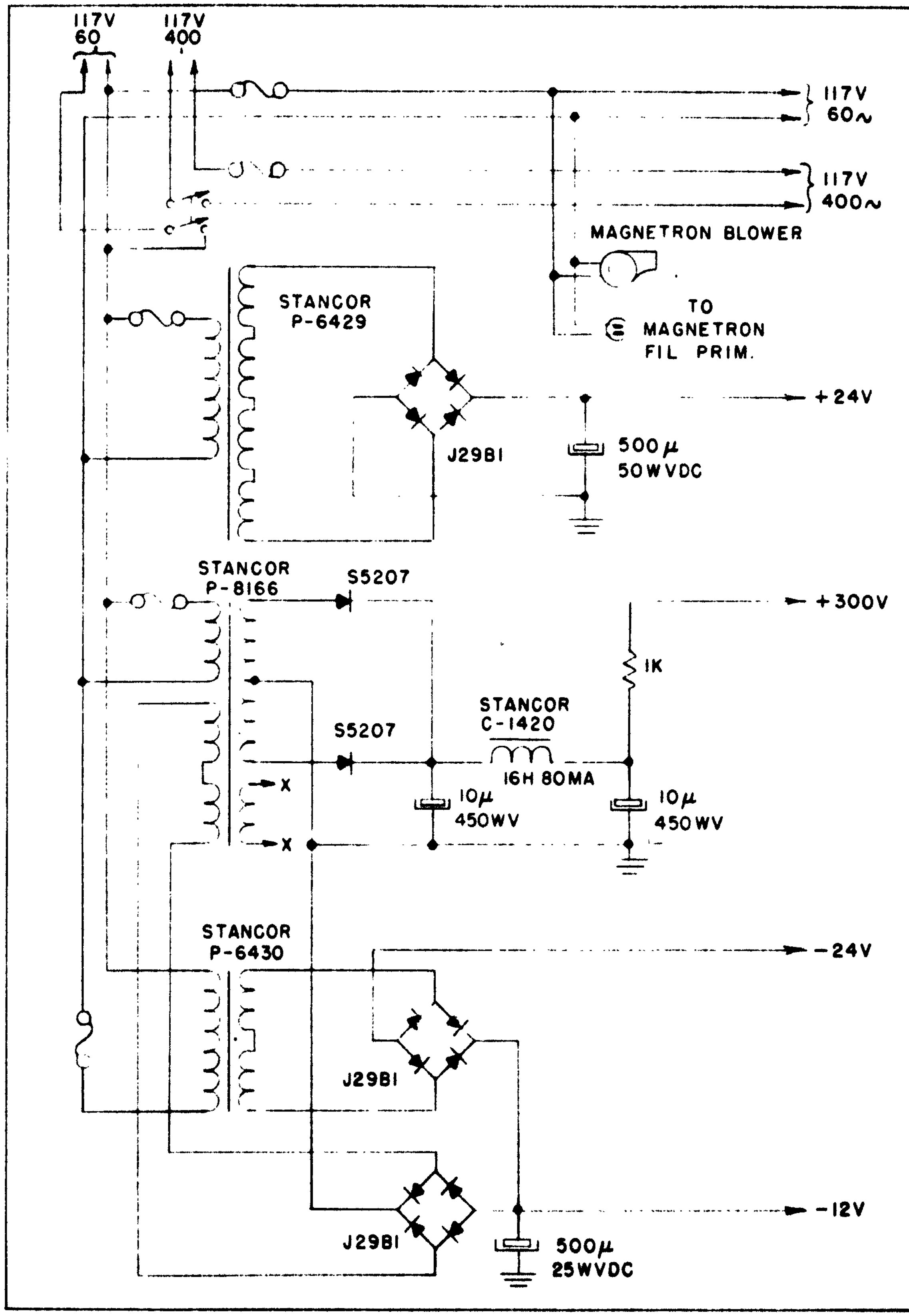

Fig. 11 - Power Supply

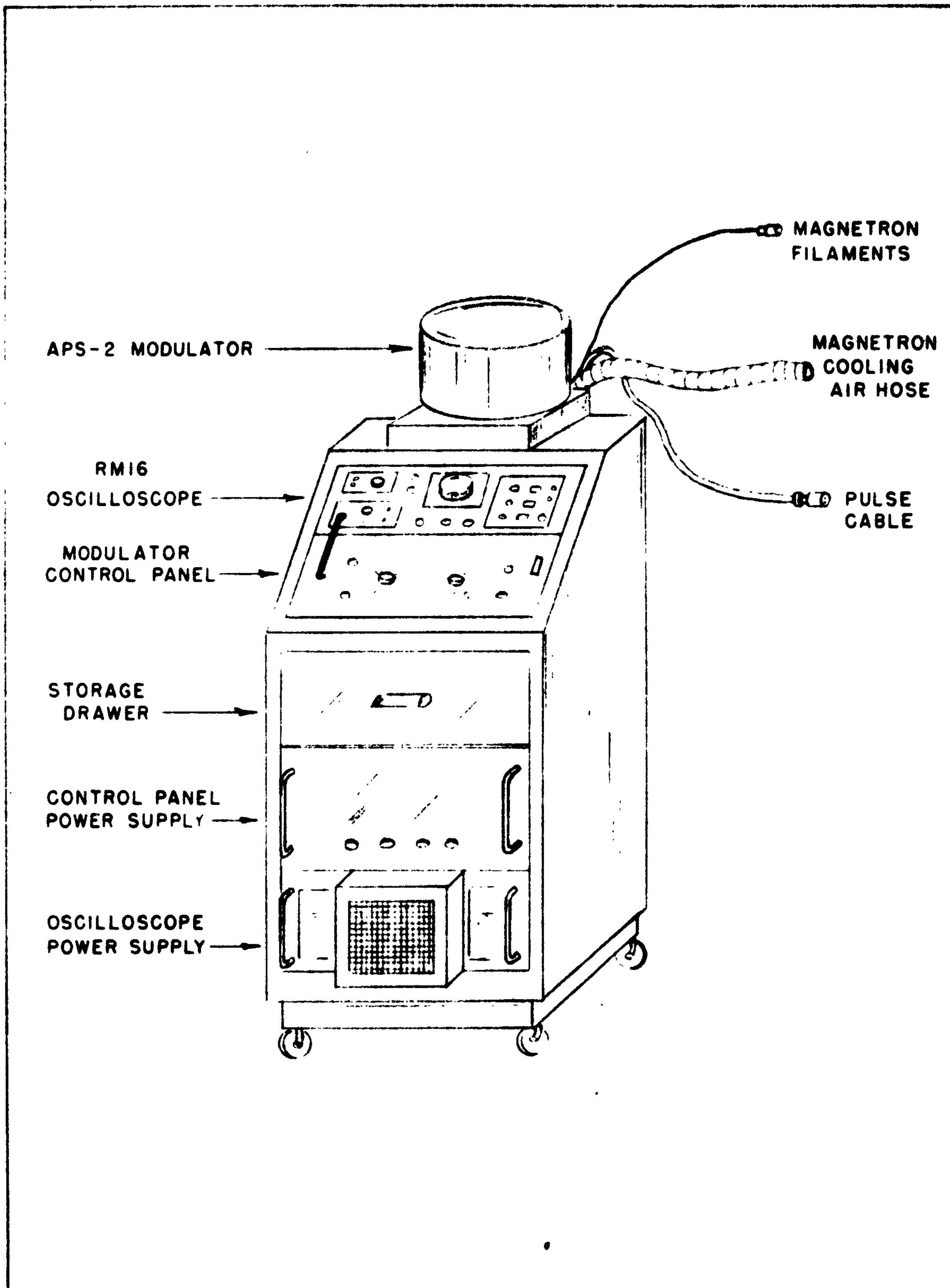

Fig. 12 - Portable Radar Modulator

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Midwest Research Institute, KCMo.
RESEARCH STUDIES RELATED TO MAPPING, GEODESY AND POSITION DETERMINATION, R.S. Brown, E.J. Martin, Jr., Final Technical Report, 19 Dec. 1961 - 18 Dec. 1962, 71 pp., 12 illus. Contract DA-44-009-ENG-3769, Project 8T35-12-001-01.

Unclassified Report

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